Review of integrals involving e^x

$$\int e^x\,dx = e^x + C \qquad \int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C \qquad \int f'(x)e^{f(x)}\,dx = e^{f(x)} + C$$

Example 22

Evaluate: **(a)** $\int_0^2 e^{2x}\,dx$ **(b)** $\int_{-0.5}^{1.5}\left(e^x - e^{-x}\right)dx$

Solution

(a) $\int_0^2 e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_0^2$

$= \frac{1}{2}\left(e^4 - e^0\right)$

$= \frac{e^4 - 1}{2}$ is the exact value.

$= 26.80$ using a calculator and writing the answer correct to 2 d.p.

(b) $\int_{-0.5}^{1.5}\left(e^x - e^{-x}\right)dx = \left[e^x + e^{-x}\right]_{-0.5}^{1.5}$

$= \left(e^{1.5} + e^{-1.5}\right) - \left(e^{-0.5} + e^{0.5}\right)$

$= 2.4496$ correct to 4 d.p.

It is an interesting exercise to show that the exact value of this integral is $\frac{(e+1)(e-1)^2}{e^{1.5}}$

Example 23

Calculate the area bounded by the curve $y = e^{1.5x}$, the coordinate axes and the line $x = 2$.

Solution

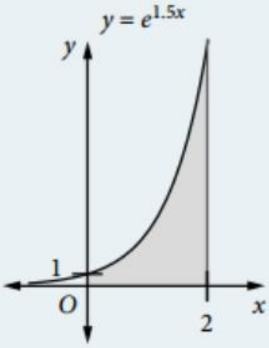
$y = e^{1.5x}$, $y = 0$, $x = 2$

Area $= \int_0^2 e^{1.5x}\,dx$

$= \left[\frac{2}{3}e^{1.5x}\right]_0^2$

$= \frac{2}{3}\left(e^3 - e^0\right)$

$= \frac{2\left(e^3 - 1\right)}{3} \approx 12.72$ units^2



Review of integrals involving logarithms

$$\int \frac{1}{x}\,dx = \log_e|x| + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|(ax+b)| + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \log_e|[f(x)]| + C$$

$$\int \frac{af'(x)}{f(x)}\,dx = a\log_e|[f(x)]| + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

Example 24

Evaluate: **(a)** $\int_1^2 \frac{3}{x+1}\,dx$ **(b)** $\int_3^4 \frac{2x-1}{x^2-x-2}\,dx$ **(c)** $\int_2^4 \frac{x^2-1}{x}\,dx$ **(d)** $\int_1^3 2^x\,dx$

Solution

(a)
$$\int_1^2 \frac{3}{x+1}\,dx = 3\int_1^2 \frac{1}{x+1}\,dx$$
$$= 3\left[\log_e(x+1)\right]_1^2$$
$$= 3(\log_e 3 - \log_e 2)$$
$$= 3\log_e 1.5$$
$$\approx 1.216$$

(b)
$$\int_3^4 \frac{2x-1}{x^2-x-2}\,dx = \left[\log_e\left(x^2-x-2\right)\right]_3^4$$
$$= \log_e 10 - \log_e 4$$
$$= \log_e 2.5$$
$$\approx 0.916$$

(c) $\frac{x^2-1}{x} = \frac{x^2}{x} - \frac{1}{x} = x - \frac{1}{x}$, so:
$$\int_2^4 \frac{x^2-1}{x}\,dx = \int_2^4\left(x - \frac{1}{x}\right)dx$$
$$= \left[\frac{x^2}{2} - \ln x\right]_2^4$$
$$= 8 - \ln 4 - (2 - \ln 2)$$
$$= 6 - \ln 2$$
$$\approx 5.307$$

(d)
$$\int_1^3 2^x\,dx = \left[\frac{2^x}{\ln 2}\right]_1^3$$
$$= \frac{1}{\ln 2}\left(2^3 - 2^1\right)$$
$$= \frac{7}{\ln 2} \approx 10.10 \ \ (2\text{ d.p.})$$

If question **(c)** had asked for an exact answer, the answer would be written as $6 - \ln 2$.

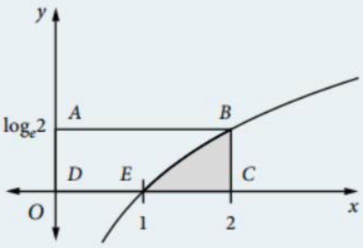
Example 25

Find the area bounded by the curve $y = \log_e x$, the x-axis and the ordinate $x = 2$.

Solution

Area $= \int_1^2 \log_e x \, dx$

You don't learn how to evaluate this integral in this course. Instead, draw a diagram to see whether there may be another way to calculate the area.



You require the area of the shaded region BCE. It can be obtained by finding the area of the rectangle $ABCD$ and subtracting the area $ABED$.

Because $y = \log_e x$, you can write $x = e^y$.

At $x = 2$, $y = \log_e 2$:

$$\begin{aligned}\text{Area } ABED &= \int_0^{\log_e 2} e^y \, dy \\ &= \left[e^y\right]_0^{\log_e 2} \\ &= e^{\log_e 2} - e^0 \\ &= 2 - 1 \\ &= 1\end{aligned}$$

Area $ABCD = 2\log_e 2$

$\therefore$ Area $BCE = 2\log_e 2 - 1$

≈ 0.386 units^2

Example 26

Differentiate $x\log_e x$. Hence find the primitive of $\log_e x$ and so evaluate $\int_1^2 \log_e x\,dx$.

Solution

Let $y = x\log_e x = uv$, where $u = x$ and $v = \log_e x$.

Product rule, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$: $\frac{d}{dx}\left(x\log_e x\right) = 1\times\log_e x + x\times\frac{1}{x}$

$$\therefore \frac{d}{dx}\left(x\log_e x\right) = \log_e x + 1$$

Rearrange: $\log_e x = \frac{d}{dx}\left(x\log_e x\right) - 1$

The primitive of the derivative of a function is the function itself, so:

$$\int \frac{d}{dx}\left(x\log_e x\right)dx = x\log_e x + C$$

Hence: $\int \log_e x\,dx = \int \frac{d}{dx}\left(x\log_e x\right)dx - \int 1\,dx$

$$= x\log_e x - x + C$$

$$\therefore \int_1^2 \log_e x\,dx = \left[x\log_e x - x\right]_1^2$$

$$= (2\log_e 2 - 2) - (\log_e 1 - 1)$$

$$= 2\log_e 2 - 1$$

$$\approx 0.386$$

This is the same answer as in Example 25, obtained by a different method.