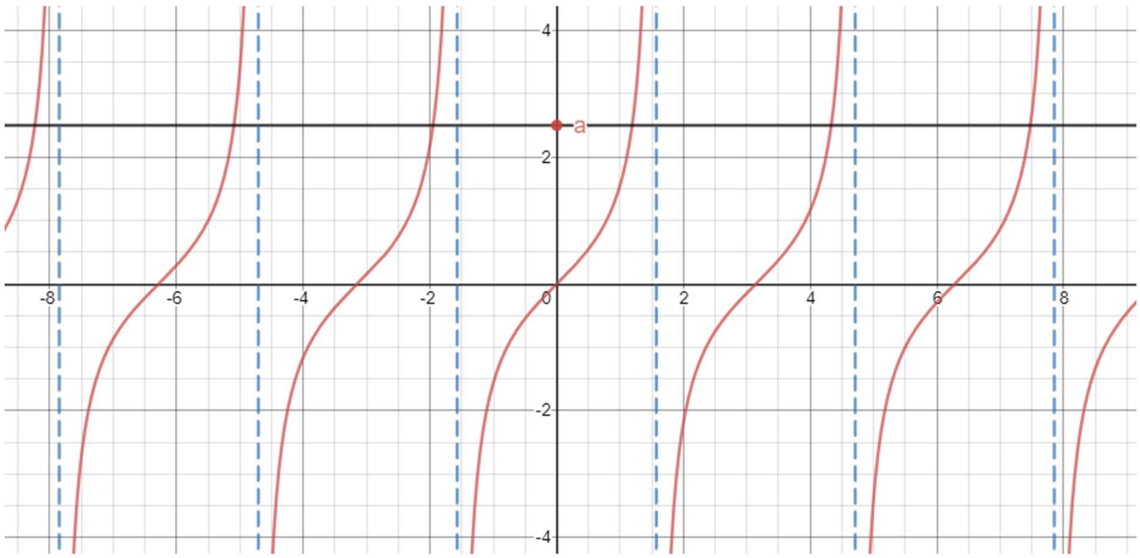
1. Equations of the form $\boldsymbol{a = \tan\theta}$ ("a" being a constant)

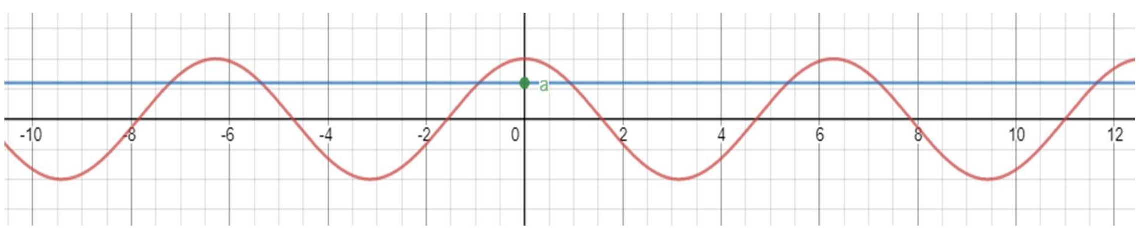


The obvious solution is $\theta = \tan^{-1} a$

But there are other solutions as *tangent* is periodic of period π

The general solution is $$\boldsymbol{\theta = (\tan^{-1} a) + n\pi}$$ where n is an integer

2. Equations of the form $\boldsymbol{a = \cos\theta}$ ("a" being a constant)



The obvious solutions are $\begin{cases} \theta = \cos^{-1} a \\ \theta = -\cos^{-1} a \end{cases}$

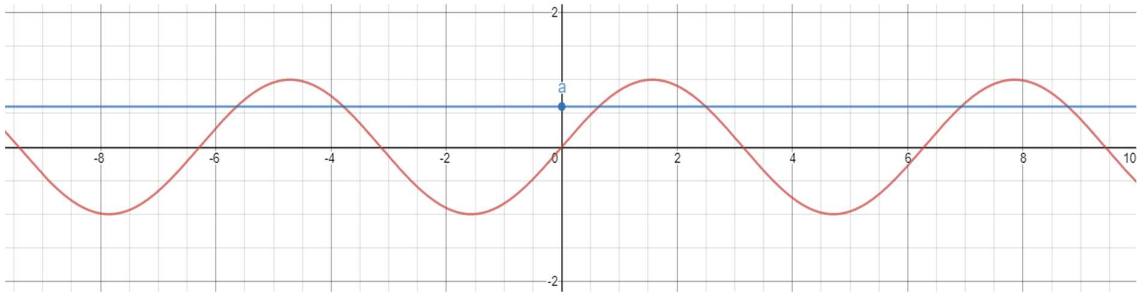
But there are other solutions as *cosine* is periodic of period 2π

Therefore all solutions are $\begin{cases} \theta = (\cos^{-1} a) + n \times 2\pi \\ \theta = (-\cos^{-1} a) + n \times 2\pi \end{cases}$ where n is an integer

or simply written $\begin{cases} \theta = \cos^{-1} a + 2n\pi \\ \theta = -\cos^{-1} a + 2n\pi \end{cases}$

which summarises as $$\boldsymbol{\theta = \pm(\cos^{-1} a) + 2n\pi}$$ where n is an integer

3. Equations of the form $\boldsymbol{a = \sin\theta}$ ("a" being a constant)



The obvious solutions are $\begin{cases} \theta = \sin^{-1} a \\ \theta = \pi - \sin^{-1} a \end{cases}$

But there are other solutions as *sine* is periodic of period 2π

Therefore all solutions are $\begin{cases} \theta = (\sin^{-1} a) + k \times 2\pi \\ \theta = (\pi - \sin^{-1} a) + k \times 2\pi \end{cases}$ where k is an integer

or simply written $\begin{cases} \theta = \sin^{-1} a + 2k\pi \\ \theta = \pi - \sin^{-1} a + 2k\pi \end{cases}$

or $\begin{cases} \theta = \sin^{-1} a + 2k\pi \\ \theta = -\sin^{-1} a + (2k+1)\pi \end{cases}$

which can also be written as $\begin{cases} \theta = \mathbf{1} \times \sin^{-1} a + 2k\pi \\ \theta = (\mathbf{-1}) \times \sin^{-1} a + (2k+1)\pi \end{cases}$

Now noting that $\mathbf{1} = (-1)^{2k}$ and that $(\mathbf{-1}) = (-1)^{2k+1}$

the equations transform as $\begin{cases} \theta = (-1)^{2k} \times \sin^{-1} a + 2k\pi \\ \theta = (-1)^{2k+1} \times \sin^{-1} a + (2k+1)\pi \end{cases}$

or $\begin{cases} \theta = (-1)^{n} \times \sin^{-1} a + n\pi & \textit{with } n\textit{" an } \textbf{\textit{even}} \textit{ integer} \\ \theta = (-1)^{n} \times \sin^{-1} a + n\pi & \textit{with "}n\textit{" an } \textbf{\textit{odd}} \textit{ integer} \end{cases}$

which summarises as $\boldsymbol{\theta = (-1)^{n} \times (\sin^{-1} a) + n\pi}$ where n is an integer

Example 9

(a) Solve, for $0 \le x \le 2\pi$, $\sqrt{2}\sin\left(\frac{x}{2}-\frac{\pi}{3}\right)=1$.

(b) Solve, for $-3\pi \le x \le 3\pi$, $\tan\left(\frac{x}{3}+\frac{\pi}{6}\right)=1$.

(c) Solve, for $-2\pi \le x \le 2\pi$, $2\cos 2\left(\frac{x}{4}-\frac{\pi}{6}\right)=\sqrt{3}$.

Solution

(a) $\sqrt{2}\sin\left(\frac{x}{2}-\frac{\pi}{3}\right)=1$

Divide by $\sqrt{2}$: $\sin\left(\frac{x}{2}-\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$

$0 \le x \le 2\pi$ so $0 \le \frac{x}{2} \le \pi$: $\frac{x}{2}-\frac{\pi}{3}=\frac{\pi}{4},\frac{3\pi}{4}$

$$\frac{x}{2}=\frac{\pi}{4}+\frac{\pi}{3},\frac{3\pi}{4}+\frac{\pi}{3}$$

$$x=\frac{7\pi}{12},\frac{13\pi}{12}$$

The only valid solution is: $x=\frac{7\pi}{12}$

(b) $\tan\left(\frac{x}{3}+\frac{\pi}{6}\right)=1$

$-3\pi \le x \le 3\pi$ so $-\pi \le \frac{x}{3} \le \pi$: $\frac{x}{3}+\frac{\pi}{6}=-\frac{3\pi}{4},\frac{\pi}{4}$

$$\frac{x}{3}=-\frac{3\pi}{4}-\frac{\pi}{6},\frac{\pi}{4}-\frac{\pi}{6}$$

$$\frac{x}{3}=-\frac{11\pi}{12},\frac{\pi}{12}$$

$$x=-\frac{11\pi}{4},\frac{\pi}{4}$$

(c) $2\cos 2\left(\frac{x}{4}-\frac{\pi}{6}\right)=\sqrt{3}$

$$2\cos\left(\frac{x}{2}-\frac{\pi}{3}\right)=\sqrt{3}$$

$$\cos\left(\frac{x}{2}-\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

$-2\pi \le x \le 2\pi$ so $-\pi \le \frac{x}{2} \le \pi$: $\frac{x}{2}-\frac{\pi}{3}=-\frac{\pi}{6},\frac{\pi}{6}$

$$\frac{x}{2}=\frac{\pi}{6},\frac{\pi}{2}$$

$$x=\frac{\pi}{3},\pi$$

Another transformation that can be used to solve equations is to square both sides of the equation. This may introduce solutions that are not valid for the original equation, so all answers need to be checked by substituting them into the original equation.

Example 10

Solve $\sin x + 2\cos x = 1$ for $0 \le x \le 2\pi$.

Solution

$\sin x + 2\cos x = 1$

Rearrange the equation: $2\cos x = 1 - \sin x$ [1]

Square both sides of [1]: $4\cos^2 x = 1 - 2\sin x + \sin^2 x$

Use the Pythagorean identity: $4(1 - \sin^2 x) = 1 - 2\sin x + \sin^2 x$

$$5\sin^2 x - 2\sin x - 3 = 0$$

Factorise: $(5\sin x + 3)(\sin x - 1) = 0$

$$\sin x = -\frac{3}{5}, 1 \quad [2]$$

$$x = \pi + 0.6435, 2\pi - 0.6435, \frac{\pi}{2}$$

$$x = 3.784, 5.640, \frac{\pi}{2}$$

To check if all these values are valid, you can substitute them into the original equation.

Alternatively, without a calculator it is easier to substitute the solutions for $\sin(x)$, as follows.

Substitute [2] (first value) into [1] to check: $2\cos x = 1 - \left(-\frac{3}{5}\right)$

$\cos x = \frac{4}{5}$

As this is positive, x must be in Q1 or Q4.

Hence the only valid answer is $x = 5.640$

Substitute [2] (second value) into [1] to check: $2\cos x = 1 - (-1)$

$\cos x = 0$

The only valid answer is $x = \frac{\pi}{2}$

The remaining answer $x = 3.785$ is not valid as $\cos(3.785) = -0.8$

Hence $x = \frac{\pi}{2}$, 5.640 are the only valid solutions.