PARTIAL FRACTIONS, LINEAR FACTORS

You will now develop the technique of partial fractions. This will be applied for use in integrating certain rational functions. In this section you may benefit from working through the numerical examples before looking at the theory.

Identity of polynomial expressions

If two polynomials of the n-th degree are equal for more than n values of the variable then they are **identically equal**, i.e. equal for all values of the variable.

You use this result to find the numerator of partial fractions.

Example 1

Express $\frac{5x+1}{(x-1)(x+2)}$ in partial fractions.

Solution

Method 1

Let $\frac{5x+1}{(x-1)(x+2)} \equiv \frac{a}{x-1} + \frac{b}{x+2}$, $x \neq 1, -2$.

Write with common denominator: $\frac{5x+1}{(x-1)(x+2)} \equiv \frac{a(x+2)+b(x-1)}{(x-1)(x+2)}$, $x \neq 1, -2$

Write the numerators: $5x+1 \equiv a(x+2)+b(x-1)$

Now use the identity property of polynomials and substitute two values for x. As we are now dealing only with the numerators, we can use $x=1$ and $x=-2$ as the two values for x.

Let $x=1$: $6 = 3a + 0$

$a = 2$

Let $x=-2$: $-9 = -3b$

$b = 3$

Hence: $\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$

Method 2

Write the numerators: $5x+1 \equiv a(x+2)+b(x-1)$

Expand and simplify RHS: $5x+1 \equiv (a+b)x + 2a - b$

Equate coefficients: $5 = a + b$ [1]

$1 = 2a - b$ [2]

[1] + [2]: $6 = 3a$ $\therefore a = 2$

Substitute into [1]: $5 = 2 + b$ $\therefore b = 3$

Hence: $\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$

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Rational functions

A rational function $f(x)$ is the ratio of two polynomials, $f(x) = \frac{A(x)}{B(x)}$, defined for all values of x except those for which $B(x) = 0$.

If the polynomial degree of $A(x) \geq$ degree of $B(x)$, then you can divide $B(x)$ into $A(x)$:

$$A(x) = B(x) \times Q(x) + R(x) \quad \text{where degree of } R(x) < \text{degree of } B(x)$$

which leads to:

$$f(x) = Q(x) + \frac{R(x)}{B(x)}$$

Consider $f(x) = \frac{x^2 + x - 1}{x^2 - 1}$. By division you find $f(x) = 1 + \frac{x}{x^2 - 1}$, so that $Q(x) = 1$, $R(x) = x$ and $B(x) = x^2 - 1$.

This means that $(x^2 + x - 1) = (x^2 - 1) \times 1 + x$.

As $x^2 - 1 = (x-1)(x+1)$, you have $B(x) = (x-1)(x+1)$.

The problem of partial fraction decomposition arises when $B(x)$ is a product of polynomials of lower degree, i.e. $B(x) = B_1(x) \times B_2(x)$ with degree $B_1(x) > 0$, degree $B_2(x) > 0$.

You wish to find polynomials $m_1(x)$, $m_2(x)$ such that: $\frac{R(x)}{B(x)} = \frac{m_1(x)}{B_1(x)} + \frac{m_2(x)}{B_2(x)}$

Now $m_1(x)$ and $m_2(x)$ can be found if: $R(x) \equiv m_1(x) \times B_2(x) + m_2(x) \times B_1(x)$

Comparison of degrees shows that you can suppose degree $m_1(x) <$ degree $B_1(x)$, degree $m_2(x) <$ degree $B_2(x)$.

Considering $\frac{R(x)}{B(x)} = \frac{x}{(x-1)(x+1)} = \frac{m_1(x)}{(x-1)} + \frac{m_2(x)}{(x+1)}$ you can write $m_1(x) = a$ and $m_2(x) = b$. These must be constants: the degree of the denominator is one, so the degree of the numerator must be zero.

Hence $\frac{R(x)}{B(x)} = \frac{x}{(x-1)(x+1)} = \frac{a}{(x-1)} + \frac{b}{(x+1)}$ and you now need to find the values of a and b.

You have:	$x \equiv a(x+1) + b(x-1)$
i.e.	$x \equiv (a+b)x + a - b$
Equate coefficients:	$(a+b) = 1$ and $a - b = 0$, $\therefore a = b$
$\therefore$	$2a = 1$, $a = 0.5$ and $b = 0.5$

Hence: $\frac{R(x)}{B(x)} = \frac{x}{(x-1)(x+1)} = \frac{0.5}{(x-1)} + \frac{0.5}{(x+1)}$

Thus you can write $f(x) = 1 + \frac{x}{x^2 - 1}$ as: $f(x) = 1 + \frac{0.5}{x-1} + \frac{0.5}{x+1}$

Linear factors

Consider the general case, where $B(x)$ is a product of distinct linear factors:

$$B(x) = k(x - a_1)(x - a_2)\dots(x - a_n)$$

You want to discover if constants $c_1, c_2, \dots c_n$ exist so that: $\frac{R(x)}{B(x)} = \frac{c_1}{x - a_1} + \frac{c_2}{x - a_2} + \dots + \frac{c_n}{x - a_n}$

Linear factors—Method 1 (equating coefficients)

Consider the monic case when $n = 2$, so that $k = 1$ and $B(x)$ is a quadratic.

$R(x) = dx + e$ and $B(x) = (x - a_1)(x - a_2)$:	$\frac{R(x)}{B(x)} = \frac{dx + e}{(x - a_1)(x - a_2)} = \frac{c_1}{x - a_1} + \frac{c_2}{x - a_2}$
Write with common denominator:	$\frac{dx + e}{(x - a_1)(x - a_2)} = \frac{c_1(x - a_2) + c_2(x - a_1)}{(x - a_1)(x - a_2)}$
Write the numerators:	$dx + e \equiv c_1(x - a_2) + c_2(x - a_1)$
Using identity property of polynomials, equate coefficients:	$d = c_1 + c_2$
	$e = -(a_2 c_1 + a_1 c_2)$

As $a_1 \neq a_2$, these two equations can be solved for c_1 and c_2.

In the general case, $a_1, a_2, \dots a_n$ are distinct, so the coefficient equations can be solved for $c_1, c_2, \dots c_n$.

This is best demonstrated with a numerical example.

Example 2

Reduce $\frac{x+1}{(x-2)(x-3)}$ to its partial fractions using linear factors method 1 (equating coefficients).

Solution

Let $\frac{x+1}{(x-2)(x-3)}=\frac{c_1}{x-2}+\frac{c_2}{x-3}$

Write with common denominator: $\frac{x+1}{(x-2)(x-3)}=\frac{c_1(x-3)+c_2(x-2)}{(x-2)(x-3)}$

Write the numerators: $x+1\equiv(c_1+c_2)x-(3c_1+2c_2)$

Using identity property, equate coefficients: $c_1+c_2=1$ [1]

$3c_1+2c_2=-1$ [2]

$2\times[1]$: $2c_1+2c_2=2$ [3]

$[2]-[3]$: $c_1=-3$

Substitute into [1]: $c_2=4$

Hence: $\frac{x+1}{(x-2)(x-3)}=\frac{4}{x-3}-\frac{3}{x-2}$

Linear factors—Method 2 (substitution)

Again consider the monic case when $n=2$, so that $k=1$ and $B(x)$ is a quadratic.

$R(x)=dx+e$ and $B(x)=(x-a_1)(x-a_2)$: $\frac{R(x)}{B(x)}=\frac{dx+e}{(x-a_1)(x-a_2)}=\frac{c_1}{x-a_1}+\frac{c_2}{x-a_2}$

Write with common denominator: $\frac{dx+e}{(x-a_1)(x-a_2)}=\frac{c_1(x-a_2)+c_2(x-a_1)}{(x-a_1)(x-a_2)}$

Write the numerators: $dx+e\equiv c_1(x-a_2)+c_2(x-a_1)$

Let $x=a_1$: $da_1+e=c_1(a_1-a_2)$

Solve for c_1: $c_1=\frac{da_1+e}{a_1-a_2}$

Let $x=a_2$: $da_2+e=c_2(a_2-a_1)$

Solve for c_2: $c_2=\frac{da_2+e}{a_2-a_1}$

You should always check your answer by writing the partial fractions over a common denominator again.

Example 3

Reduce $\frac{2x+1}{(x-2)(x+3)}$ to partial fractions using linear factors method 2 (substitution).

Solution

Let $\frac{2x+1}{(x-2)(x+3)}=\frac{c_1}{x-2}+\frac{c_2}{x+3}$

Write with common denominator: $\frac{2x+1}{(x-2)(x+3)}=\frac{c_1(x+3)+c_2(x-2)}{(x-2)(x+3)}$

Write the numerators: $2x+1\equiv c_1(x+3)+c_2(x-2)$

Let $x=2$: $5=5c_1+0$

$c_1=1$

Let $x=-3$: $-5=0-5c_2$

$c_2=1$

Hence: $\frac{2x+1}{(x-2)(x+3)}=\frac{1}{x-2}+\frac{1}{x+3}$

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Linear factors—Method 3 (limits)

Multiply through the expression $\frac{R(x)}{B(x)}$ and note that $B(a_1) = 0$. Hence $B(x) = B(x) - B(a_1)$.

Thus $\frac{R(x)}{B(x)} = \frac{c_1}{x - a_1} + \frac{c_2}{x - a_2} + \ldots + \frac{c_n}{x - a_n}$ becomes: $\frac{R(x)(x - a_1)}{B(x) - B(a_1)} = c_1 + \frac{c_2(x - a_1)}{x - a_2} + \ldots + \frac{c_n(x - a_1)}{x - a_n}$

Let $x \to a_1$ and recall that $\lim_{x \to a_1} \frac{x - a_1}{B(x) - B(a_1)} = \frac{1}{B'(a_1)}$ (from the definition of differentiation from first principles).

You have: RHS $\to c_1$

LHS $\to \frac{R(a_1)}{B'(a_1)}$

Repeating this process for each of the other linear factors allows the other c_i to be found.
Remember that $B(x) = k(x - a_1)(x - a_2)\ldots(x - a_n)$, i.e. each factor is monic.

Example 4

Reduce $\frac{3x - 4}{(x + 2)(x - 3)}$ to its partial fractions using linear factors method 3 (limits).

Solution

Let $\frac{3x - 4}{(x + 2)(x - 3)} = \frac{c_1}{x + 2} + \frac{c_2}{x - 3}$

$R(x) = 3x - 4 \qquad B(x) = x^2 - x - 6 \qquad B'(x) = 2x - 1$

$c_1 = \frac{R(-2)}{B'(-2)} = \frac{-10}{-4 - 1} = 2 \qquad c_2 = \frac{R(3)}{B'(3)} = \frac{5}{6 - 1} = 1$

Hence: $\frac{3x - 4}{(x + 2)(x - 3)} = \frac{2}{x + 2} + \frac{1}{x - 3}$

You should try using each of these methods to find partial fractions. You may find that you end up preferring one of the methods, but it is best to be able to use whichever is the most efficient method for a question.

Example 5

Reduce $\frac{3x}{(2x - 1)(x + 1)}$ to its partial fractions.

Solution

Let $\frac{3x}{(2x - 1)(x + 1)} = \frac{a}{2x - 1} + \frac{b}{x + 1}$

Write with common denominator: $\frac{3x}{(2x - 1)(x + 1)} = \frac{a(x + 1) + b(2x - 1)}{(2x - 1)(x + 1)}$

Write the numerators: $3x \equiv a(x + 1) + b(2x - 1)$

To find a, let $x = \frac{1}{2}$: $\frac{3}{2} = a \times \frac{3}{2} + 0$

$a = 1$

To find b, let $x = -1$: $-3 = 0 + b \times (-3)$

$b = 1$

Hence: $\frac{3x}{(2x - 1)(x + 1)} = \frac{1}{2x - 1} + \frac{1}{x + 1}$

Example 6

Express $\dfrac{54}{(x^2+x-20)(x-1)}$ using partial fractions.

Solution

Factorise the quadratic expression: $x^2+x-20=(x+5)(x-4)$

$\therefore \quad \dfrac{54}{(x+5)(x-4)(x-1)}=\dfrac{c_1}{x+5}+\dfrac{c_2}{x-4}+\dfrac{c_3}{x-1}$

Write with common denominator: $\dfrac{54}{(x+5)(x-4)(x-1)}=\dfrac{c_1(x-4)(x-1)+c_2(x+5)(x-1)+c_3(x+5)(x-4)}{(x+5)(x-4)(x-1)}$

Write the numerators: $54 \equiv c_1(x-4)(x-1)+c_2(x+5)(x-1)+c_3(x+5)(x-4)$

Let $x=-5$: $54=54c_1 \quad \therefore c_1=1$

Let $x=4$: $54=27c_2 \quad \therefore c_2=2$

Let $x=1$: $54=-18c_3 \quad \therefore c_3=-3$

Hence: $\dfrac{54}{(x^2+x-20)(x-1)}=\dfrac{1}{x+5}+\dfrac{2}{x-4}-\dfrac{3}{x-1}$

Example 7

Reduce $\dfrac{x^2}{x^2+3x+2}$ to its partial fractions.

Solution

As degree of numerator = degree of denominator, first divide by the denominator.

As a quicker way of doing the division, consider rewriting:

$$\frac{x^2}{x^2+3x+2}=\frac{(x^2+3x+2)-(3x+2)}{x^2+3x+2}=1-\frac{3x+2}{x^2+3x+2}$$

We now need to find c_1 and c_2 for $\dfrac{3x+2}{x^2+3x+2}=\dfrac{c_1}{x+1}+\dfrac{c_2}{x+2}$.

$R(x)=3x+2 \qquad B(x)=x^2+3x+2 \qquad B'(x)=2x+3$

$x=-1$: $c_1=\dfrac{R(-1)}{B'(-1)}=\dfrac{-3+2}{-2+3}=-1 \qquad x=-2$: $c_2=\dfrac{R(-2)}{B'(-2)}=\dfrac{-6+2}{-4+3}=4$

Hence: $\dfrac{3x+2}{x^2+3x+2}=\dfrac{-1}{x+1}+\dfrac{4}{x+2} \quad \therefore \dfrac{x^2}{x^2+3x+2}=1+\dfrac{1}{x+1}-\dfrac{4}{x+2}$