POLYNOMIALS - CHAPTER REVIEW

1 Perform the following polynomial divisions.

(a) $(x^3 + 2x^2 - 3x + 4) \div (x - 1)$ **(b)** $(4x^4 - 6x^2 + 10x - 40) \div (x + 3)$

a)

$$\begin{array}{r|l} & x^2 + 3x \\ x-1 & x^3 + 2x^2 - 3x + 4 \\ & x^3 - x^2 \\ \hline & 0 \quad 3x^2 - 3x + 4 \\ & \quad 3x^2 - 3x \\ \hline & \quad\quad 0 \quad 4 \end{array}$$

so $x^3 + 2x^2 - 3x + 4 = (x-1)(x^2+3x) + 4$

b)

$$\begin{array}{r|l} & 4x^3 - 12x^2 + 30x - 80 \\ x+3 & 4x^4 - 6x^2 + 10x - 40 \\ & 4x^4 + 12x^3 \\ \hline & 0 - 12x^3 - 6x^2 + 10x - 40 \\ & -12x^3 - 36x^2 \\ \hline & 0 \quad 30x^2 + 10x - 40 \\ & 30x^2 + 90x \\ \hline & -80x - 40 \\ & -80x - 240 \\ \hline & 200 \end{array}$$

So

$$4x^4 - 6x^2 + 10x - 40 = (x+3)(4x^3 - 12x^2 + 30x - 80) + 200$$

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2 Use the remainder theorem to find the remainder of the following.

(a) $x^3 - 4x^2 + 3x - 5$ divided by $(x - 2)$ **(b)** $x^4 + x^3 - 5x^2 + 4x - 2$ divided by $(x + 1)$

a) $P(2) = 8 - 16 + 6 - 5 = -7$

b) $P(-1) = (-1)^4 + (-1)^3 - 5\times 1 - 4 - 2 = -11$

3 Use the factor theorem to find the linear factors (over the rational number field) of each polynomial.

(b) $x^3 + 7x^2 + 14x + 8$ **(c)** $x^3 + 5x^2 - x - 5$

b) $P(-2) = -8 + 28 - 28 + 8 = 0$

so $(x+2)$ divides the polynomial

$$\begin{array}{r|l} & x^2 + 5x + 4 \\ x+2 & x^3 + 7x^2 + 14x + 8 \\ & x^3 + 2x^2 \\ \hline & 5x^2 + 14x + 8 \\ & 5x^2 + 10x \\ \hline & 4x + 8 \end{array}$$

$\Delta = 25 - 4\times 4 = 9 = 3^2$

$x_1 = \frac{-5+3}{2} = -1$ $x_2 = \frac{-5-3}{2} = -4$

$P(x) = (x+2)(x+1)(x+4)$

c) $P(1) = 1^3 + 5 - 1 - 5 = 0$

so $(x-1)$ divides the polynomial

$$\begin{array}{r|l} & x^2 + 6x + 5 \\ x-1 & x^3 + 5x^2 - x - 5 \\ & x^3 - x^2 \\ \hline & 6x^2 - x - 5 \\ & 6x^2 - 6x \\ \hline & 5x - 5 \end{array}$$

$\Delta = 36 - 4\times 5 = 16 = 4^2$

$x_1 = \frac{-6+4}{2} = -1$

$x_2 = \frac{-6-4}{2} = -5$

so $P(x) = (x-1)(x+1)(x+5)$

4 Let $P(x) = (x-1)(x+2)Q(x) + ax + b$, where $Q(x)$ is a polynomial and a and b are real numbers. The polynomial $P(x)$ has a factor of $x+2$. When $P(x)$ is divided by $x-1$ the remainder is 6.

(a) Find the values of a and b.

(b) Find the remainder when $P(x)$ is divided by $(x-1)(x+2)$.

a) $P(-2) = 0$ $\quad P(-2) = -2a + b = 0$ $\quad$ so $b = 2a$ ①

$P(1) = 6$ $\quad P(1) = a + b = 6$ ②

So $a + 2a = 6$ (solving ① and ② by substitution)

so $\boxed{a = 2}$ and $\boxed{b = 4}$

b) The remainder is $2x + 4$

5 Let $P(x) = x^3 + ax^2 - x + 1$ be a polynomial where a is a real number. When $P(x)$ is divided by $x - 2$ the remainder is 15. Find the remainder when $P(x)$ is divided by $x + 3$.

$P(2) = 15$ $\qquad P(2) = 8 + 4a - 2 + 1 = 15$

so $\quad 7 + 4a = 15$

$4a = 8 \qquad \boxed{a = 2}$

$P(x) = x^3 + 2x^2 - x + 1$

$$\begin{array}{r|l} & x^2 - x + 2 \\ x+3 & x^3 + 2x^2 - x + 1 \\ & x^3 + 3x^2 \\ \hline & -x^2 - x + 1 \\ & -x^2 - 3x \\ \hline & 2x + 1 \\ & 2x + 6 \\ \hline & -5 \end{array}$$

so

$$\frac{(x^3 + 2x^2 - x + 1)}{(x+3)} = x^2 - x + 2 - \frac{5}{x+3}$$

OR FASTER: using the remainder theorem

$P(-3) = (-3)^3 + 2(-3)^2 - (-3) + 1$

$P(-3) = -5$

7 The polynomial $P(x)$ is given by $P(x) = ax^3 + 15x^2 + cx - 72$, where a and c are constants. The three zeros of $P(x)$ are -3, 2 and α. Find the value of α.

$P(-3) = P(2) = P(\alpha) = 0$

We know b and d.

$P(-3) = -27a + 135 - 3c - 72 = 0$

so $-27a - 3c = -63$

$9a + c = 21$ Equation ①

$P(2) = 8a + 60 + 2c - 72 = 0$

so $8a + 2c = 12$

$4a + c = 6$ Equation ②

By elimination $5a = 15$ so $\boxed{a = 3}$

so $c = 6 - 4a = 6 - 4 \times 3 = -6$ $\boxed{c = -6}$

$P(x) = 3x^3 + 15x^2 - 6x - 72$

$P(x) = 3(x^3 + 5x^2 - 2x - 24)$

$P(x) = 3(x+3)(x^2 + 2x - 8)$

$P(x) = 3(x+3)(x-2)(x+4)$

$\boxed{\alpha = -4}$

8 The cubic polynomial $P(x) = x^3 + bx^2 + cx + d$ (where b, c, d are real numbers) has three real zeros: -1, α and $-\alpha$.

(a) Find the value of b. **(b)** Find the value of $c - d$.

a) $\alpha + \beta + \gamma = -\frac{b}{a} = -b$ so $-1 + \alpha - \alpha = -b$

$\therefore b = 1$

$P(x) = x^3 + x^2 + cx + d$

b) $P(-1) = 0$ so $-1 + 1 - c + d = 0$

$\therefore c - d = 0$

9 The polynomial $P(x) = x^3 - 4x^2 + kx + 12$ has zeros α, β, γ.

(a) Find the value of $\alpha + \beta + \gamma$. **(b)** Find the value of $\alpha\beta\gamma$.

(c) Two of the three zeros are equal in magnitude but opposite in sign. Find the third zero and hence find the value of k.

a) $\alpha + \beta + \gamma = \frac{+4}{1} = 4$

b) $\alpha\beta\gamma = \frac{-12}{1} = -12$

c) Say $\beta = -\alpha$ $\quad \therefore \gamma = 4$

$\therefore -\alpha^2 \times 4 = -12$

$\therefore \alpha^2 = 3$

so $\alpha = \sqrt{3}$ and $\beta = -\sqrt{3}$

then, knowing $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = k$.

$\therefore -3 + 4\sqrt{3} - 4\sqrt{3} = k$

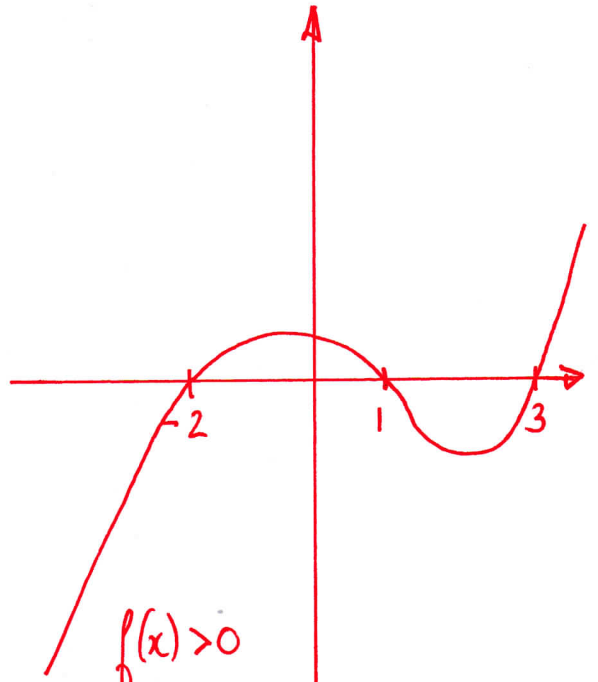
$\therefore \boxed{k = -3}$

$P(x) = x^3 - 4x^2 - 3x + 12$

POLYNOMIALS - CHAPTER REVIEW

10 Sketch graphs of each function. For what values of x is each function positive?

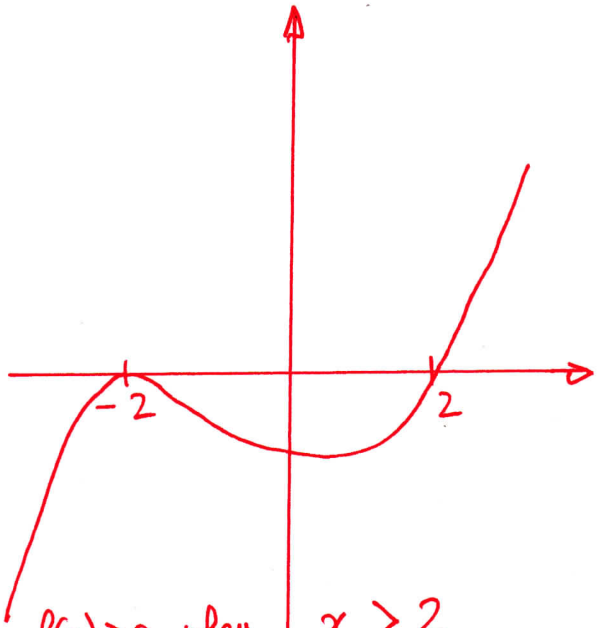
(a) $y = (x-1)(x+2)(x-3)$



$f(x) > 0$

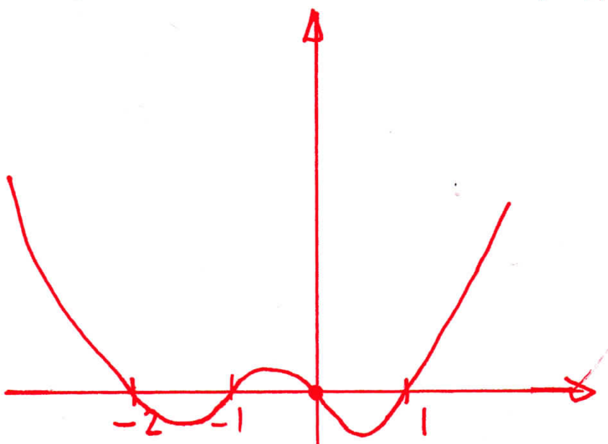
when $-2 < x < 1$ and $x > 3$

(b) $y = (x-2)(x+2)^2$



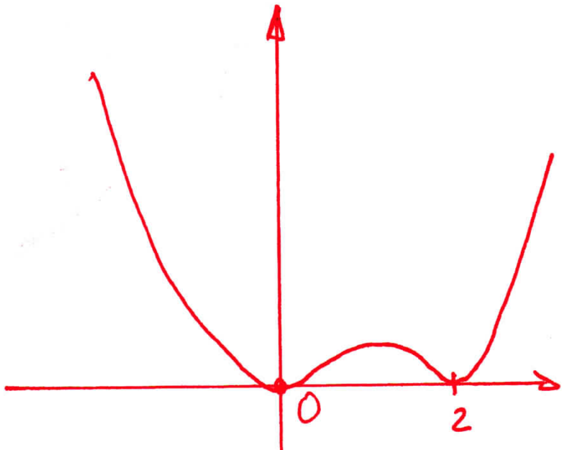
$f(x) > 0$ when $x > 2$

(c) $y = x(x^2-1)(x+2) = x(x-1)(x+1)(x+2)$



$f(x) > 0$ when
$x < -2$
$x > 1$
$-1 < x < 0$

(d) $y = x^2(x-2)^2$



$f(x) > 0$ when $x \neq 0$ and $x \neq 2$

$f(x) \geq 0$ everywhere.