DIRECTION FIELDS

Qualitative (or graphical) methods of solution

Qualitative methods are a set of graphical methods to describe the general behaviour of the solution to a differential equation without solving the equation.

Recall that $\frac{dy}{dx}$ is the slope of the curve at any point (x, y). A differential equation, such as $\frac{dy}{dx} = f(x, y)$, can be thought of as a definition of the values of the slope of the tangent to the solution curve for possible values of x and y. This enables us to sketch the graphical features of the solution. The graph showing the gradient at different points is called the **direction field** or the **slope field**.

Direction field construction on a rectangular grid

This method involves the following two steps:

1. Evaluate the derivative for a carefully selected set of points (x, y).
2. At each point (x, y), draw a short line segment of slope $\frac{dy}{dx}$.

Example 7

Construct the slope field of $\frac{dy}{dx} = xy$ on the grid:

$(0,0)$ $(0,1)$ $(0,2)$ $(0,3)$
$(1,0)$ $(1,1)$ $(1,2)$ $(1,3)$
$(2,0)$ $(2,1)$ $(2,2)$ $(2,3)$
$(3,0)$ $(3,1)$ $(3,2)$ $(3,3)$

Solution

$\frac{dy}{dx} = f(x, y) = xy$ is evaluated for each point using integer values for x and y.

For example, at the point (0, 0), $\frac{dy}{dx} = xy = 0 \times 0 = 0$.

Therefore, if the curve goes through (0, 0), its gradient at that point will be 0.

At the point (2, 3), $\frac{dy}{dx} = xy = 2 \times 3 = 6$.

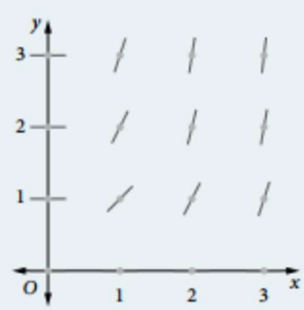
Therefore, if the curve goes through (2, 3), its gradient at that point will be 6.

All the gradients are calculated.

$f(0,0) = 0$ $f(0,1) = 0$ $f(0,2) = 0$ $f(0,3) = 0$
$f(1,0) = 0$ $f(1,1) = 1$ $f(1,2) = 2$ $f(1,3) = 3$
$f(2,0) = 0$ $f(2,1) = 2$ $f(2,2) = 4$ $f(2,3) = 6$
$f(3,0) = 0$ $f(3,1) = 3$ $f(3,2) = 6$ $f(3,3) = 9$

At each such point (x, y) on the grid, tangent segments of slope $\frac{dy}{dx} = f(x, y)$ are drawn using rise over run.

Having constructed a direction field, the short sloping lines can be used as a guide to draw smooth curves with the same gradients. These curves represent possible graphs generated by the differential equation. In some cases, more slopes may need to be drawn.

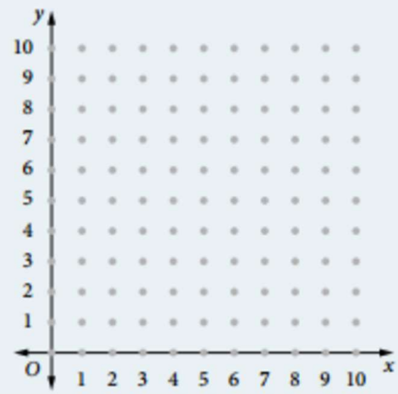


Example 8

Construct the slope field of $\frac{dy}{dx} = -2(y-5)$ on a suitable grid for $0 \le x \le 10$ and $0 \le y \le 10$.

Solution

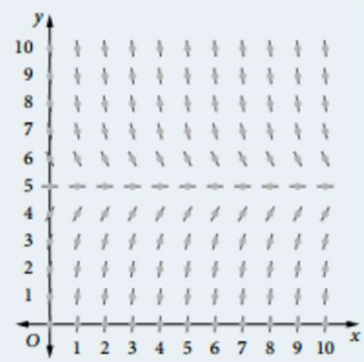
Step 1: A grid is constructed to cover the given intervals:



Step 2: At each such point (x, y) on the grid, tangent segments of the slope function are drawn. As the derivative is a function of y only, then given any specific value of y, it will be the same for all values of x. This means it only needs to be calculated for each value of y.

For $f(x, y) = -2(y - 5)$, $f(x, 0) = 10$, $f(x, 1) = 8$, $f(x, 2) = 6$, $f(x, 3) = 4$, $f(x, 4) = 2$, $f(x, 5) = 0$, $f(x, 6) = -2$, $f(x, 7) = -4$, $f(x, 8) = -6$, $f(x, 9) = -8$, $f(x, 10) = -10$.

This information is shown in the diagram at right.



Example 9

(a) Construct the slope field of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ for $-3 \le x \le 3$ and $-3 \le y \le 3$.

(b) Use the slope field to draw possible solutions to $\frac{dy}{dx} = -\frac{x}{y}$.

(c) Draw the specific solution if the curve passes through the point: **(i)** (0, 2) **(ii)** (2, −2).

(d) Suggest a possible equation of the general curve and test your answer by differentiation.

Solution

(a) **Step 1**: The differential equation is used to find the gradient at each point.

$(-3, -3)$: $\frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{-3} = -1$

$(-3, -2)$: $\frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{-2} = -1.5$

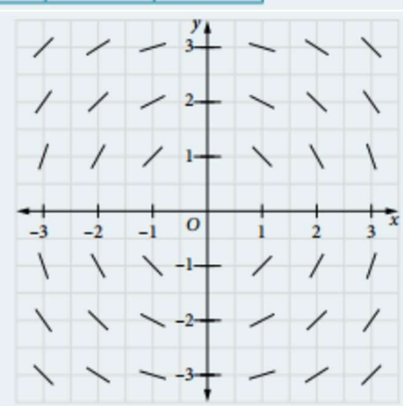
$(-3, -1)$: $\frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{-1} = -3$

$(-3, 0)$: $\frac{dy}{dx} = -\frac{x}{y} = -\frac{-3}{0}$. This is undefined, but it may mean that if the curve goes through (−3, 0), it will be vertical.

Continuing similarly for positive *x* values gives the following gradient values:

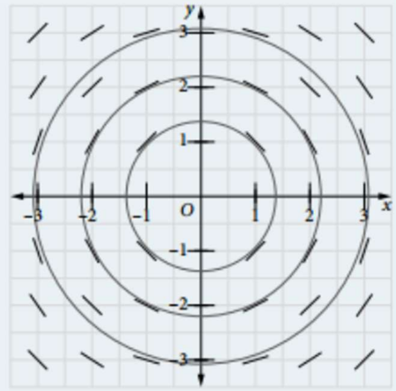
		x values						
		−3	−2	−1	0	1	2	3
y values	−3	−1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
	−2	−1.5	−1	−0.5	0	0.5	1	1.5
	−1	−3	−2	−1	0	1	2	3
	0	–	–	–	–	–	–	–
	1	3	2	1	0	−1	−2	−3
	2	1.5	1	0.5	0	−0.5	−1	−1.5
	3	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$	−1

Step 2: Coordinate axes can be used to represent the slope using a short line at each point, estimating the slope using rise over run.

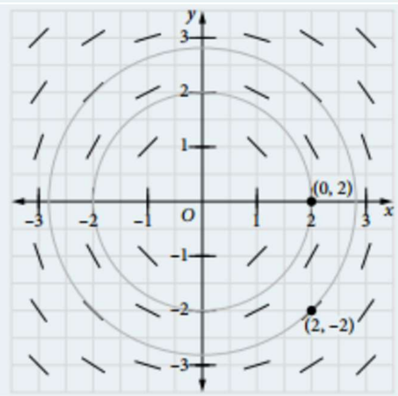


(b) The curve may be circular. Some curves can be drawn using the slopes as a guide, possibly with a compass. This diagram reinforces that the undefined slopes are at points where the curves are vertical, except for the point (0, 0). These slopes have been added to the diagram.

Circles seem to fit very well. At this point there are an infinite number of solutions to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.



(c) A circle is drawn through each given point, as shown.



(d) The equation of a circle is $x^2 + y^2 = r^2$.

(i) In this case, $r = 2$, so the equation is $x^2 + y^2 = 4$.

If a function is required, $y = \sqrt{4 - x^2}$. The positive square root is taken because the curve goes through (0, 2).

(ii) Substitute $x = 2, y = -2$ in $x^2 + y^2 = r^2$.

$2^2 + (-2)^2 = r^2 \therefore r = \sqrt{8}$, so the equation is $x^2 + y^2 = 8$.

If a function is required, $y = -\sqrt{8 - x^2}$. The negative square root is taken because the curve goes through $(2, -2)$.

The general result is tested by differentiation.

Method 1

Need to differentiate each term of $x^2 + y^2 = r^2$ with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(r^2)$$

Using the chain rule,

$$\frac{d}{dx}(x^2) + \frac{d}{dy}(y^2) \times \frac{dy}{dx} = \frac{d}{dx}(r^2)$$

Remembering that r^2 is a constant:

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

Method 2

Need to differentiate both sides of $y = \sqrt{r^2 - x^2}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{r^2 - x^2}\right)$$

Let $u = r^2 - x^2$, so $\frac{du}{dx} = -2x$

$$\frac{dy}{dx} = \frac{d}{du}\left(\sqrt{u}\right) \times \frac{du}{dx}$$

$$= \frac{d}{du}\left(u^{\frac{1}{2}}\right) \times \frac{du}{dx}$$

$$= \frac{1}{2}u^{-\frac{1}{2}} \times (-2x)$$

$$= -\frac{x}{\sqrt{u}}$$

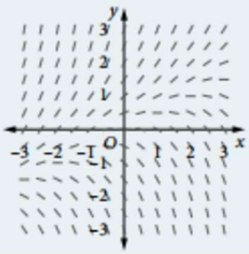
However, $y = \sqrt{r^2 - x^2} = \sqrt{u}$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

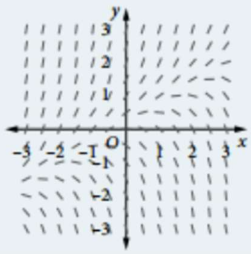
Example 10

Which of the following direction fields could have the differential equation $\frac{dy}{dx} = x + ky$ as a solution if $k > 0$?

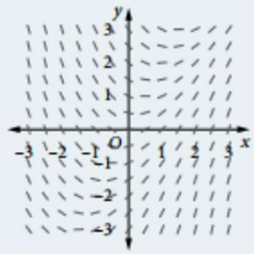
A



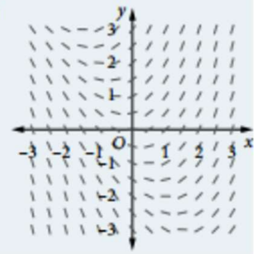
B



C



D



Solution

The behaviour of the derivative is considered for different values of x and y.

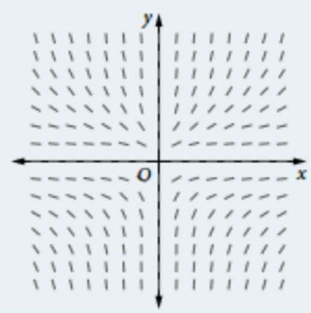
At $x = 0$ (the y-axis), $\frac{dy}{dx} = ky$, so $\frac{dy}{dx}$ will be positive where y is positive and negative where y is negative. This eliminates C.

If $x > 0$ and $y > 0$, $\frac{dy}{dx}$ will be positive and increases as x increases. $\frac{dy}{dx}$ also increases as y increases. This eliminates A and B.

Further analysis will show that D is consistent with $\frac{dy}{dx} = x + ky$, $k > 0$.

Example 11

The slope field for a differential equation is shown:



Which of the following could be the form of the differential equation represented by this slope field, if $k > 0$?

A $\frac{dy}{dx} = kxy^2$ **B** $\frac{dy}{dx} = kxy^3$

C $\frac{dy}{dx} = \frac{ky^2}{x}$ **D** $\frac{dy}{dx} = \frac{ky^3}{x}$

Solution

Consider how the slope lines are changing and whether this is consistent with each proposed option.

By examining option A, if $x > 0$ and $y > 0$, the slope is positive and if x is constant (any horizontal row), so the slope should increase as x increases. However, if a horizontal row is examined, the slope decreases as x increases. This also rules out option B.

Alternatively, the gradient in the slope field is undefined where $x = 0$, yet the derivatives in options A and B are defined and zero where $x = 0$.

For options C and D, if $x > 0$ and $y > 0$ the slope is expected to be positive. This is consistent with the slope field. The gradients are undefined where $x = 0$, which is also consistent with the slope field.

The difference between these two options is the power of y.

If $x > 0$ and $y > 0$, the slope should increase as y increases, which can be seen by examining any vertical column.

If $x > 0$ and $y < 0$, the slope in C will be positive and increase as y increases, but the slope in D will be negative and decrease (become more negative, i.e. steeper) as y increases. By looking at the fourth quadrant, it can be seen that option C correctly predicts the behaviour of the slope.