SOLVING QUADRATIC EQUATIONS WITH COMPLEX COEFFICIENTS

1 Find:

(a) $\sqrt{2i}$ (b) $\sqrt{3+4i}$ (c) $\sqrt{5-12i}$ (d) $\sqrt{-8+15i}$ (e) $\sqrt{-3-4i}$ (f) $\sqrt{1+i}$

a) we look for the complex number $a+ib$ such that $(a+ib)^2 = 2i$

$(a+ib)^2 = a^2-b^2+2iab$ So $\begin{cases} a^2-b^2=0 \\ 2ab=2 \end{cases} \iff \begin{cases} a^2=b^2 \\ ab=1 \end{cases}$

So either $a=\pm b$

if $a=b$, then $a^2=1$ so $a=\pm 1$, and $b=\pm 1$

if $a=-b$ then $a^2=-1$ which is impossible as a is a real number.

So two solutions: $\sqrt{2i}=1+i$ or $\sqrt{2i}=-1-i$

Check: $(1+i)^2 = 1-1+2i = 2i$

$(-1-i)^2 = (-1)^2+(-i)^2+2i = 2i$ indeed.

b) we look for the complex number $a+ib$ such that $(a+ib)^2=3+4i$

So $a^2-b^2+2iab = 3+4i$ So $\begin{cases} a^2-b^2=3 \\ 2ab=4 \end{cases} \iff \begin{cases} a^2-b^2=3 \\ ab=2 \end{cases}$

$(a^2+b^2)^2 = (a^2-b^2)^2+4a^2b^2 = 3^2+4\times 2^2 = 25$

So $a^2+b^2=5$

$\therefore 2a^2=8$ $\therefore a^2=4$ $\therefore a=\pm 2$

if $a=2$ then $b=1$ and $\sqrt{3+4i}=2+i$

if $a=-2$ then $b=-1$ and $\sqrt{3+4i}=-2-i$

two solutions $(2+i)$ and $(-2-i)$

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c) We look for the complex number $a+ib$ such that $(a+ib)^2 = 5-12i$

$\Leftrightarrow a^2 - b^2 + 2iab = 5 - 12i$ so $\begin{cases} a^2 - b^2 = 5 \\ ab = -6 \end{cases}$

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2 = 25 + 4 \times 36 = 169 = 13^2$

so $a^2 + b^2 = 13$ $\therefore$ $2a^2 = 18$ $\therefore$ $a^2 = 9$ $\therefore$ $a = \pm 3$

if $a = 3$ then $b = -2$ $\therefore$ $\sqrt{5-12i} = 3a - 2i$

if $a = -3$ then $b = 2$ $\therefore$ $\sqrt{5-12i} = -3 + 2i$

two solutions $(3a - 2i)$ and $(-3+2i)$

d) we look for the complex number $(a+ib)$ such that $(a+ib)^2 = -8+15i$

$\Leftrightarrow \begin{cases} a^2 - b^2 = -8 \\ 2ab = 15 \end{cases}$ $\Leftrightarrow \begin{cases} a^2 + b^2 = -8 \\ ab = 15/2 \end{cases}$

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2 = 64 + 4 \times \left(\frac{15}{2}\right)^2 = 289 = 17^2$

So $a^2 + b^2 = 15$ $\therefore$ $2a^2 = 9$ $\therefore$ $a^2 = \frac{9}{2}$ $\therefore$ $a = \pm \frac{3}{\sqrt{2}}$

if $a = +\frac{3}{\sqrt{2}}$ then $b = \frac{15}{2} \times \frac{\sqrt{2}}{3} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

if $a = -\frac{3}{\sqrt{2}}$ then $b = \frac{15}{2} \times \left(-\frac{\sqrt{2}}{3}\right) = -\frac{5\sqrt{2}}{2}$

So two solutions $\left(\frac{3}{\sqrt{2}} + \frac{5i\sqrt{2}}{2}\right)$ or $\left(\frac{-3}{\sqrt{2}} - \frac{5i\sqrt{2}}{2}\right)$

e) we look for the complex number $(a+ib)$ such that $(a+ib)^2 = -3-4i$

$\Leftrightarrow a^2 - b^2 + 2iab = -3-4i \Leftrightarrow \begin{cases} a^2 - b^2 = -3 \\ ab = -2 \end{cases}$

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2 = 9 + 4\times(-2)^2 = 25 = 5^2$

So $a^2+b^2 = 5$ $\therefore 2a^2 = 2$ $\therefore a = \pm 1$

if $a = 1$ then $b = -2$ $\quad z = 1-2i$

if $a = -1$ then $b = 2$ $\quad z = -1+2i$

two solutions $(1-2i)$ and $(-1+2i)$.

f) we look for the complex number $(a+ib)$ such that $(a+ib)^2 = 1+i$

$\Leftrightarrow \begin{cases} a^2 - b^2 = 1 \\ 2ab = 1 \end{cases}$

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2 = 1^2 + 4\times\left(\frac{1}{2}\right)^2 = 2$

So $a^2+b^2 = \sqrt{2}$ $\therefore 2a^2 = 1+\sqrt{2}$ $\therefore a^2 = \frac{1+\sqrt{2}}{2}$

if $a = \sqrt{\frac{1+\sqrt{2}}{2}}$ then $b = \frac{1}{2}\frac{1}{\sqrt{\frac{\sqrt{2}+1}{2}}} = \frac{1}{\sqrt{2}\sqrt{\sqrt{2}+1}}$

if $a = -\sqrt{\frac{1+\sqrt{2}}{2}}$ then $b = -\frac{1}{\sqrt{2}\sqrt{\sqrt{2}+1}}$

two solutions $\left(\sqrt{\frac{1+\sqrt{2}}{2}} + \frac{i}{\sqrt{2}\sqrt{\sqrt{2}+1}}\right)$ and $\left(-\sqrt{\frac{1+\sqrt{2}}{2}} - \frac{i}{\sqrt{2}\sqrt{\sqrt{2}+1}}\right)$

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2 Solve the following statements.

(a) $x^2 + 2x + 2i = 0$ (b) $x^2 - 4x + 2 - i = 0$ (c) $x^2 + 2(2+i)x + 3 = 0$

a) $\Delta = 2^2 - 4 \times 2i = 4 - 8i$

we look for the complex number $(a+ib)$ such that $(a+ib)^2 = 4 - 8i$

$$\Rightarrow \begin{cases} a^2 - b^2 = 4 \\ 2ab = -8 \end{cases} \Leftrightarrow \begin{cases} a^2 - b^2 = 4 \\ ab = -4 \end{cases}$$

$$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2 = 16 + 4 \times (-4)^2 = 5 \times 16 = (4\sqrt{5})^2$$

So $a^2 + b^2 = 4\sqrt{5}$ $\therefore 2a^2 = 4 + 4\sqrt{5}$ $\therefore a^2 = 2 + 2\sqrt{5}$

So $a = \sqrt{2+2\sqrt{5}}$ and then $b = \dfrac{-4}{\sqrt{2+2\sqrt{5}}}$

OR $a = -\sqrt{2+2\sqrt{5}}$ then $b = \dfrac{4}{\sqrt{2+2\sqrt{5}}}$

$$x = \frac{-2 \pm \left[\sqrt{2+2\sqrt{5}} - \dfrac{4i}{\sqrt{2+2\sqrt{5}}}\right]}{2}$$

b) $\Delta = (-4)^2 - 4(2-i) = 16 - 8 + 4i = 8 + 4i$

we look for the complex number $(a+ib)$ such that $(a+ib)^2 = 8 + 4i$

$$\Rightarrow \begin{cases} a^2 - b^2 = 8 \\ 2ab = 4 \end{cases} \Rightarrow \begin{cases} a^2 - b^2 = 8 \\ ab = 2 \end{cases}$$

$$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2 = 64 + 4 \times 4 = 80 = [4\sqrt{5}]^2$$

So $a^2 + b^2 = 4\sqrt{5}$ $\therefore 2a^2 = 8 + 4\sqrt{5}$

$\therefore a^2 = 4 + 2\sqrt{5}$

$a = \pm\sqrt{4+2\sqrt{5}}$

ii) $a = \sqrt{4+2\sqrt{5}}$ Then $b = \dfrac{2}{\sqrt{4+2\sqrt{5}}}$

two solutions: $x = \dfrac{4 \pm \left[\sqrt{4+2\sqrt{5}} + \dfrac{2i}{\sqrt{4+2\sqrt{5}}}\right]}{2}$

c) $\Delta = [2(2+i)]^2 - 4 \times 3 = [4+2i]^2 - 12 = 16 - 4 + 16i - 12$

So $\Delta = 16i = (4\sqrt{i})^2 = \left[4\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\right]^2$ (result from Q1 a)).

So two solutions: $x = \dfrac{-2(2+i) \pm \left[4\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\right]}{2}$

i.e. $x = -(2+i) \pm 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$x = -2 - i \pm \left[\sqrt{2} + i\sqrt{2}\right]$

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(g) $ix^2 + 2ix + 3 = 0$ **(h)** $(2-i)x^2 + 2x + 1 = 0$

g) $ix^2 + 2ix + 3 = 0 \iff x^2 + 2x + \frac{3}{i} = 0$

but $\frac{1}{i} = \frac{i}{i^2} = -i$ $\iff x^2 + 2x - 3i = 0$

$\iff (x+1)^2 - 1 = 3i$

$\iff (x+1)^2 = 3i - 1$ we need to find the root of $(3i-1)$

i.e. $(a+ib)^2 = 3i - 1$

$\begin{cases} a^2 - b^2 = -1 \\ 2iab = 3i \end{cases}$ $\begin{cases} a^2 - b^2 = -1 \\ ab = 3/2 \end{cases}$

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2 = (-1)^2 + 4\left(\frac{3}{2}\right)^2 = 1 + 9 = 10$

So $a^2 + b^2 = \sqrt{10}$ then $2a^2 = \sqrt{10} - 1$ $a^2 = \frac{\sqrt{10}-1}{2}$

As $a^2 - b^2 = -1$

if $a = \sqrt{\frac{\sqrt{10}-1}{2}}$ then $b = \frac{3}{2}\sqrt{\frac{2}{\sqrt{10}-1}} = \frac{3}{\sqrt{2}\sqrt{\sqrt{10}-1}}$

if $a = -\sqrt{\frac{\sqrt{10}-1}{2}}$ then $b = -\frac{3}{2}\sqrt{\frac{2}{\sqrt{10}-1}} = \frac{-3}{\sqrt{2}\sqrt{\sqrt{10}-1}}$

So $x + 1 = \pm\left[\sqrt{\frac{\sqrt{10}-1}{2}} + \frac{3i}{\sqrt{2}\sqrt{\sqrt{10}-1}}\right]$ Solution ①

$x = -1 \pm \left[\sqrt{\frac{\sqrt{10}-1}{2}} + \frac{3i}{\sqrt{2}\sqrt{\sqrt{10}-1}}\right]$

b) $(2-i)x^2 + 2x - 1 = 0$

$\Delta = 4 - 4 \times (2-i) = -4 + 4i$

to find $\sqrt{\Delta}$, we look for the complex number $(a+ib)$ such that

$(a+ib)^2 = -4+4i \implies \begin{cases} a^2 - b^2 = -4 \\ 2ab = 4 \end{cases} \implies \begin{cases} a^2 - b^2 = -4 \\ ab = -2 \end{cases}$

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2 = (-4)^2 + 4 \times 4 = 32 = (4\sqrt{2})^2$

so $a^2 + b^2 = 4\sqrt{2}$ $\quad \therefore 2a^2 = 4\sqrt{2} - 4$

$\therefore a^2 = 2\sqrt{2} - 2$ $\quad \therefore a = \pm\sqrt{[2\sqrt{2}-2]}$.

if $a = 2\sqrt{2} - 2$ then $b = \dfrac{-2}{2\sqrt{2}-2} = \dfrac{-1}{\sqrt{2}-1}$

i.e $b = \dfrac{-[\sqrt{2}+1]}{2-1} = -\sqrt{2} - 1$

$\therefore x = \dfrac{-2 \pm \left[2\sqrt{2}-2 - i(\sqrt{2}+1)\right]}{2(2-i)}$

$x = \dfrac{-1}{2-i} \pm \left[\dfrac{2\sqrt{2}-2 - i(\sqrt{2}+1)}{2(2-i)}\right]$

$x = \dfrac{1}{i-2} \pm \left[\dfrac{\sqrt{2}-1}{2-i} - \dfrac{i(\sqrt{2}+1)}{2(2-i)}\right]$

can be simplified further ...

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3 (a) Expand and simplify the expression $(x-3)(x-1-i)(x-1+i)$.

(b) Hence, or otherwise, solve the equation $x^3 - 5x^2 + 8x = 6$.

a) $(x-3)(x-1-i)(x-1+i) = (x-3)(x^2 - x + \cancel{ix} - x + 1 \cancel{-i} \cancel{-ix} + \cancel{i} + 1)$

$= (x-3)(x^2 - 2x + 2)$

$= x^3 - 2x^2 + 2x - 3x^2 + 6x - 6$

$= x^3 - 5x^2 + 8x - 6$

b) So $x^3 - 5x^2 + 8x = 6 \iff x^3 - 5x^2 + 8x - 6 = 0$

$\iff (x-3)(x-1-i)(x-1+i) = 0$

$(x-3)(x-(1+i))(x-(1-i)) = 0$

So $x = 3$

or $x = 1+i$

or $x = 1-i$