DEFINITE INTEGRALS AND SUBSTITUTION

When using a substitution to evaluate a definite integral you must take care with the limits of integration. The original limits are for values for x, but after substitution the variable will become u (or some other new variable), so the limits similarly need to become values for u (or the other new variable). To do this, substitute the limits into the change-of-variable equation to find the limits for the new variable.

Example 12

Evaluate: **(a)** $\int_1^2 2x\sqrt{x^2-1}\,dx$ using the substitution $u = x^2 - 1$

(b) $\int_{-5}^3 x\sqrt{4-x}\,dx$ using the substitution $u = 4 - x$

(c) $\int_0^1 x^2\left(x^3+1\right)^4 dx$ using the substitution $u = x^3 + 1$.

Solution

(a) $u = x^2 - 1, \frac{du}{dx} = 2x$

Limits: for $x = 1, u = 1^2 - 1 = 0$

for $x = 2, u = 2^2 - 1 = 3$

$$\int_1^2 2x\sqrt{x^2-1}\,dx = \int_0^3 \sqrt{u} \times \frac{du}{dx} dx$$

$$= \int_0^3 u^{\frac{1}{2}}\,du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^3$$

$$= \frac{2}{3}\left(3^{\frac{3}{2}} - 0\right) = 2\sqrt{3}$$

(b) $u = 4 - x, \frac{du}{dx} = -1$

$x = 4 - u$, so $x\sqrt{4-x} = (4-u)\sqrt{u}$

or $-x\sqrt{4-x} = (u-4)\sqrt{u}$

Limits: for $x = -5, u = 4 + 5 = 9$

for $x = 3, u = 4 - 3 = 1$

$$\int_{-5}^3 x\sqrt{4-x}\,dx = \int_9^1 (u-4)\sqrt{u} \times \frac{du}{dx} dx$$

$$= \int_9^1 \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right) du$$

$$= \left[\frac{2}{5}u^{\frac{5}{2}} - 4 \times \frac{2}{3}u^{\frac{3}{2}}\right]_9^1$$

$$= \left(\frac{2}{5} - \frac{8}{3}\right) - \left(\frac{2}{5} \times 9^{\frac{5}{2}} - \frac{8}{3} \times 9^{\frac{3}{2}}\right)$$

$$= \frac{2}{5} - \frac{8}{3} - \frac{2}{5} \times 3^5 + \frac{8}{3} \times 3^3$$

$$= -\frac{412}{15}$$

(c) $u = x^3 + 1, \frac{du}{dx} = 3x^2$

Limits: for $x = 0, u = 1$

for $x = 1, u = 1^3 + 1 = 2$

$$\int_0^1 x^2\left(x^3+1\right)^4 dx = \frac{1}{3}\int_1^2 u^4 \times \frac{du}{dx} dx$$

$$= \frac{1}{3}\int_1^2 u^4\,du$$

$$= \frac{1}{3}\left[\frac{1}{5}u^5\right]_1^2$$

$$= \frac{1}{15}\left[u^5\right]_1^2$$

$$= \frac{1}{15}(32-1) = \frac{31}{15}$$

Useful result

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ If you reverse the limits of integration, you change the sign of the integral.

In Example **11(b)** the integral $\int_9^1\left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$ could have been written as $-\int_1^9\left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$.

Check that this integral gives the same answer.

Useful result

You may have noticed that for an integral of the form $\int f'(x)(f(x))^n \, dx$, using the substitution $u = f(x)$ gives $\frac{du}{dx} = f'(x)$ so that the integral becomes: $\int f'(x)(f(x))^n \, dx = \int u^n \, du$

$$= \frac{1}{n+1}u^{n+1} + C = \frac{1}{n+1}(f(x))^{n+1} + C$$

If you can recognise that $\int f'(x)(f(x))^n \, dx = \frac{1}{n+1}(f(x))^{n+1} + C$, you can obtain your answer faster. This result will be especially useful in section 11.6 (page 250) when integrating powers of trigonometric functions.