1 Express each of the following in the form $r\sin(x+\alpha)$.

(a) $\sin x + \cos x$ $= a\sin x + b\cos x$	**(c)** $5\sin x + 12\cos x$, $0° < \alpha < 90°$
$r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$	$r = \sqrt{5^2+12^2} = \sqrt{169} = 13$
so $\sin x + \cos x = \sqrt{2}\sin(x+\alpha)$	so $5\sin x + 12\cos x = 13\sin(x+\alpha)$
—— $= \sqrt{2}[\sin x\cos\alpha + \cos x\sin\alpha]$	—— $= 13[\sin x\cos\alpha + \cos x\sin\alpha]$
So $\sqrt{2}\cos\alpha = 1$	So $13\cos\alpha = 5$
and $\sqrt{2}\sin\alpha = 1$	and $13\sin\alpha = 12$
(by equating the coefficients of $\sin x$ and $\cos x$)	(by equating the coefficients of $\sin x$ and $\cos x$)
$\cos\alpha = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	so $\cos\alpha = 5/13$
$\sin\alpha = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	and $\sin\alpha = 12/13$
so $\alpha = \frac{\pi}{4}$	So $\alpha \simeq 67.38°$
$\sin x + \cos x = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$	$5\sin x + 12\cos x = 13\sin(x + 67.38°)$

2 Express each of the following in the form $r\sin(x-\alpha)$.

(a) $\sin x - \sqrt{3}\cos x = a\sin x - b\cos x$

$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$

So:

$\sin x - \sqrt{3}\cos x = 2\sin(x-\alpha)$

$\text{———} = 2[\sin x\cos\alpha - \cos x\sin\alpha]$

By equating the coefficients of $\sin x$ and $\cos x$, we obtain:

$$\begin{cases} 2\cos\alpha = 1 \\ -2\sin\alpha = -\sqrt{3} \end{cases}$$

OR $\begin{cases} \cos\alpha = 1/2 \\ \sin\alpha = \sqrt{3}/2 \end{cases}$

So $\alpha = \pi/3$

$\sin x - \sqrt{3}\cos x = 2\sin\left(x - \frac{\pi}{3}\right)$

(c) $2\sin x - \cos x$, $0° < \alpha < 90°$

$r = \sqrt{2^2 + 1^2} = \sqrt{5}$

So

$2\sin x - \cos x = \sqrt{5}\sin(x-\alpha)$

$\text{———} = \sqrt{5}[\sin x\cos\alpha - \cos x\sin\alpha]$

By equating the coefficients of $\sin x$ and $\cos x$, we get:

$$\begin{cases} \sqrt{5}\cos\alpha = 2 \\ -\sqrt{5}\sin\alpha = -1 \end{cases}$$

OR $\begin{cases} \cos\alpha = 2/\sqrt{5} \\ \sin\alpha = 1/\sqrt{5} \end{cases}$

$\alpha \simeq 26.56°$

$2\sin x - \cos x = \sqrt{5}\sin(x - 26.56°)$

3 Express each of the following in the form $r\cos(x-\alpha)$.

(a) $\cos x + \sin x = a\cos x + b\sin x$

$r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$

So

$\cos x + \sin x = \sqrt{2}\cos(x-\alpha)$

$\text{——} = \sqrt{2}\left[\cos x\cos\alpha + \sin x\sin\alpha\right]$

By equating the coefficients of $\cos x$ and $\sin x$, we obtain:

$$\begin{cases}\sqrt{2}\cos\alpha = 1\\ \sqrt{2}\sin\alpha = 1\end{cases}$$

OR $\begin{cases}\cos\alpha = 1/\sqrt{2} = \sqrt{2}/2\\ \sin\alpha = 1/\sqrt{2} = \sqrt{2}/2\end{cases}$

$\alpha = \pi/4$

$\cos x + \sin x = \sqrt{2}\cos(x - \pi/4)$

METHOD 2 $\quad r=\sqrt{2} \quad a = 1$ and $b = 1$

$\tan\alpha = \dfrac{b}{a} = \dfrac{1}{1} = 1$

so $\alpha = \pi/4$

$\cos x + \sin x = \sqrt{2}\cos(x - \pi/4)$

(d) $3\cos x + 2\sin x,\ 0° < \alpha < 90°$

$r = \sqrt{a^2+b^2} = \sqrt{3^2+2^2} = \sqrt{13}$

So

$3\cos x + 2\sin x = \sqrt{3}\cos(x-\alpha)$

$\text{——} = \sqrt{3}\left[\cos x\cos\alpha + \sin x\sin\alpha\right]$

By equating the coefficients of $\cos x$ and $\sin x$, we obtain:

$$\begin{cases}\sqrt{13}\cos\alpha = 3\\ \sqrt{13}\sin\alpha = 2\end{cases}$$

OR $\begin{cases}\cos\alpha = 3/\sqrt{13}\\ \sin\alpha = 2/\sqrt{13}\end{cases}$

$\alpha \simeq 33° \, 41'$

$3\cos x + 2\sin x = \sqrt{13}\cos(x - 33°41')$

5 Which expression is equivalent to $8\sin x - 15\cos x$?

A $17\cos(x - 61°56')$ **B** $17\sin(x - 61°56')$

C $17\cos(x + 61°56')$ **D** $17\sin(x + 61°56')$

We try any value ($x = 0$ for example) $8\sin 0 - 15\cos 0 = -15$

So cannot be $\boxed{A}$ (as it's >0) nor $\boxed{C}$, nor $\boxed{D}$ (both positive when $x=0$)

So it has to be $\boxed{B}$

6 Find **(i)** the maximum and **(ii)** the minimum value of the following expressions. Also find the smallest positive values of x for which the maximum and minimum occur.

(a) $\sin x - \sqrt{3}\cos x = E$

we try to put the expression in the form $r\sin(x+\alpha)$

$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

The maximum value is $+2$ and the minimum value is -2

$\sin x - \sqrt{3}\cos x = 2\sin(x+\alpha)$

$\quad = 2[\sin x\cos\alpha + \cos x\sin\alpha]$

So by equating the coefficients of $\sin x$ and $\cos x$ on both sides, we obtain $\begin{cases} 2\cos\alpha = 1 \\ 2\sin\alpha = -\sqrt{3}\end{cases}$

$\begin{cases}\cos\alpha = 1/2 \\ \sin\alpha = -\sqrt{3}/2\end{cases}$ $\quad \alpha = -\pi/3$

$E = 2\sin\left(x - \pi/3\right)$

So maximum occurs when $\sin\left(x - \frac{\pi}{3}\right) = 1$

or $x - \frac{\pi}{3} = \frac{\pi}{2}$ i.e $x = \frac{5\pi}{6}$

and minimum when $x - \pi/3 = -\pi/2$

$x = -\frac{\pi}{6}$ $\left(\text{or } x = \frac{11\pi}{6}\right)$

(c) $2\sqrt{3}\cos x - 2\sin x = E$

again, we try to express as $r\sin(x+\alpha)$

$r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12+4} = \sqrt{16} = 4$

Maximum value is 4, minimum is (-4)

$2\sqrt{3}\cos x - 2\sin x = 4\sin(x+\alpha)$

$\quad = 4[\sin x\cos\alpha + \cos x\sin\alpha]$

By equating the coefficients of $\sin x$ and $\cos x$, we obtain:

$\begin{cases}4\cos\alpha = -2 \\ 4\sin\alpha = 2\sqrt{3}\end{cases}$

$\begin{cases}\cos\alpha = -1/2 \\ \sin\alpha = \sqrt{3}/2\end{cases}$ $\quad \alpha = \frac{2\pi}{3}$

$E = 4\sin\left(x + \frac{2\pi}{3}\right)$

Maximum is when $x + \frac{2\pi}{3} = \frac{\pi}{2}$

$\left(x = -\frac{\pi}{6} \text{ or } \frac{11\pi}{6}\right)$

Minimum when $x + \frac{2\pi}{3} = -\frac{\pi}{2}$

$x = -\frac{7\pi}{6}$ or $5\pi/6$

i.e $x = \frac{5\pi}{6}$ (minimum value)

7 Solve:

(a) $\cos x + \sin x = 1, 0 \le x \le 2\pi$

we are going to put the LHS as $r\sin(x+\alpha)$.

$r = \sqrt{1^2+1^2} = \sqrt{2}$, so

$\iff \sqrt{2}\left[\frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x\right] = 1$

$\iff \sqrt{2}\left[\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x\right] = 1$

$\iff \sqrt{2}\sin\left(x+\frac{\pi}{4}\right) = 1$

$\iff \sin\left(x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \sin\frac{\pi}{4}$

the general solution of this equation is $x+\frac{\pi}{4} = (-1)^n\frac{\pi}{4} + n\pi$ (n integer).

$\iff x = (-1)^n\frac{\pi}{4} - \frac{\pi}{4} + n\pi$

$n=0$ gives: $x=0$ which is an acceptable solution

$n=1$ gives: $x = -\frac{\pi}{4}\times 2 + \pi = \frac{\pi}{2}$ which is also acceptable.

$n=2$ gives: $x = \frac{\pi}{4} - \frac{\pi}{4} + 2\pi = 2\pi$ which is also acceptable.

Other values of n give values outside the interval $[0, 2\pi]$.

So 3 solutions: $0, \frac{\pi}{2}$ and 2π

(h) $\cos x - \sin x = -1, -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

we transform the LHS as $r\sin(x+\alpha)$

$r = \sqrt{1^2+(-1)^2} = \sqrt{2}$, so

$\iff \sqrt{2}\left[\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x\right] = -1$

$\iff \sin\frac{3\pi}{4}\cos x + \cos\frac{3\pi}{4}\sin x = -\frac{1}{\sqrt{2}}$

$\iff \sin\left(x+\frac{3\pi}{4}\right) = \sin\frac{5\pi}{4}$

General solution is:

$x+\frac{3\pi}{4} = (-1)^n \times \frac{5\pi}{4} + n\pi$ where n is an integer.

$\iff x = (-1)^n\frac{5\pi}{4} - \frac{3\pi}{4} + n\pi$

$n=0$ gives: $x = \frac{5\pi}{4} - \frac{3\pi}{4} = \frac{\pi}{2}$ which is an acceptable solution

$n=1$ gives: $x = -\frac{5\pi}{4} - \frac{3\pi}{4} + \pi = -\pi$ which is not an acceptable solution.

$n=2$ gives: $x = \frac{5\pi}{4} - \frac{3\pi}{4} + 2\pi$ No good.

$n=-1$ gives: $x = -\frac{5\pi}{4} - \frac{3\pi}{4} - \pi$ No good

$n=3$ gives: $x = -\frac{5\pi}{4} - \frac{3\pi}{4} + 3\pi = \pi$ which is acceptable.

Other values of n gives values of x outside $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

So 2 solutions: $\pi/2$ and π

8 Sketch the graph of $f(x) = \sqrt{3}\cos x - \sin x$, $0 \le x \le 2\pi$. Use your sketch to find the values of x for which:

(a) $f(x) = 1$ (b) $f(x) > 1$

We transform $f(x)$ as $r\sin(x+\alpha)$

$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$

$\sqrt{3}\cos x - \sin x = 2\sin(x+\alpha)$

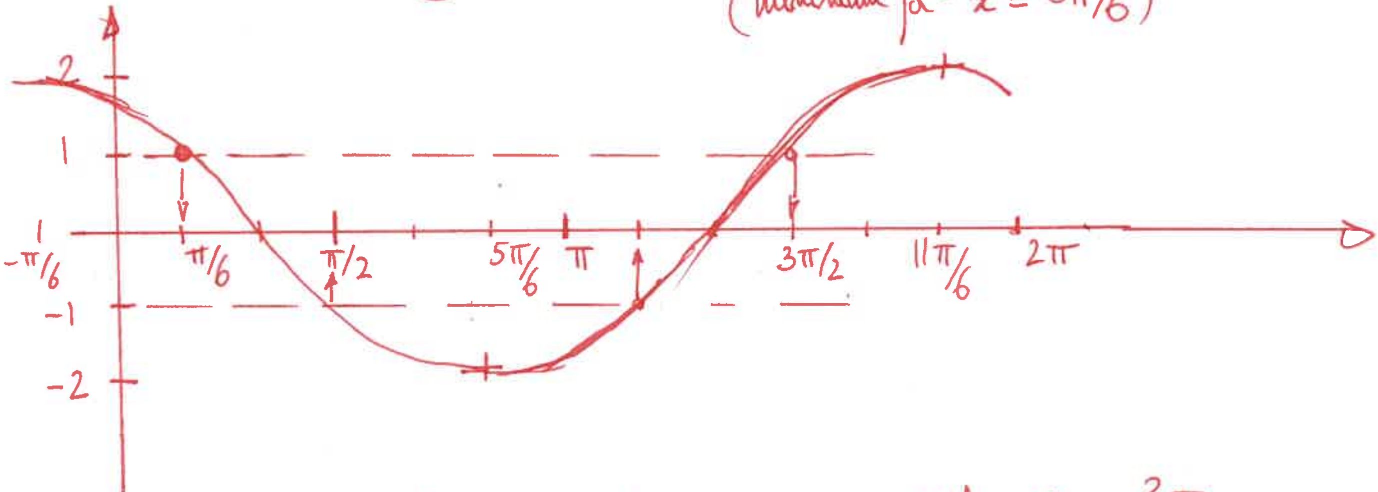
$\_\_\_\_\_\_ = 2[\sin x \cos\alpha + \cos x \sin\alpha]$

So $\begin{cases} 2\cos\alpha = -1 \\ 2\sin\alpha = \sqrt{3} \end{cases}$ OR $\begin{cases} \cos\alpha = -\frac{1}{2} \\ \sin\alpha = \sqrt{3}/2 \end{cases}$ $\alpha = \frac{2\pi}{3}$

So $\sqrt{3}\cos x - \sin x = 2\sin\left(x + \frac{2\pi}{3}\right)$.

$f(x)$ varies between (-2) and $(+2)$ and is maximum for the 1st time when $x + \frac{2\pi}{3} = \frac{\pi}{2}$ i.e. $x = -\frac{\pi}{6}$ $\left(\text{or } x = \frac{11\pi}{6}\right)$.

(minimum for $x = 5\pi/6$)



It looks like $f(x) = 1$ for $x = \frac{\pi}{6}$ and $x = \frac{3\pi}{2}$

indeed $f\left(\frac{\pi}{6}\right) = \sqrt{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{6} = \sqrt{3} \times \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{3}{2} - \frac{1}{2} = 1$

$f\left(\frac{3\pi}{2}\right) = \sqrt{3}\cos\left(\frac{3\pi}{2}\right) - \sin\left(\frac{3\pi}{2}\right) = \sqrt{3} \times 0 - (-1) = 1$

For $f(x) = -1$ $x = \frac{\pi}{2}$ and $x = \frac{7\pi}{6}$

indeed $f\left(\frac{\pi}{2}\right) = \sqrt{3}\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = 0 - 1 = -1$

and $f\left(\frac{7\pi}{6}\right) = \sqrt{3}\cos\left(\frac{7\pi}{6}\right) - \sin\left(\frac{7\pi}{6}\right) = \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2}\right) = -\frac{3}{2} + \frac{1}{2} = -1$

YES!