MATHEMATICAL REPRESENTATION OF MOTION IN PHYSICAL TERMS

Mechanics is a topic that includes both kinematics and dynamics.

- **Kinematics** is the study of the motion of bodies without reference to the causes of their motion.
- **Dynamics** is the study of the effects of forces that cause bodies at rest to move, or that cause bodies in motion to have their state of motion altered.

Forces produce accelerations, so you will be using the principles of kinematics in much of the dynamics material of this chapter.

In this topic, all bodies are treated as particles, which means that all forces are regarded as acting through a single point in the body (as if the body were just a single point). Hence the terms 'particle', 'object' and 'body' will be interchangeable.

Newton's first law of motion

Consider a book resting on the top of your desk. It will remain there in that state of rest unless some external force or forces are applied to change that state. The book is unable to alter its state of rest by itself.

Similarly, consider a marble rolling along a smooth horizontal floor at a constant speed. By itself, this marble is unable to increase or decrease its speed or to change its direction.

This property of bodies is summed up in Newton's first law of motion:

A body remains at rest or in uniform motion in a straight line unless it is acted on by a non-zero resultant force.

A body can be acted on by several forces that balance each other, so that the resultant force is zero. (For example, a book resting on a horizontal desk is acted on by two forces: the weight force of gravity that acts vertically downwards and the reaction force of the desk that acts vertically upwards.) A zero resultant force is the equivalent of no force acting, so the body remains stationary or in its original state of motion.

Newton's second law of motion

Experience suggests that a given force will produce different accelerations in different bodies. For example, on a smooth horizontal floor the same amount of rolling force applied to a marble and to a heavy steel ball would produce a larger acceleration in the marble. The steel ball is more massive than the marble. Similarly, a piano is more massive than a school desk, as is shown by the fact that it is easier to make the desk move than it is to get the piano to move.

This property of bodies that determines their response to an applied force is called their **inertial mass**. Inertial mass is a measure of a body's resistance to acceleration. This mass is essentially related to the amount of matter that makes up the body. It can be shown experimentally that the ratio of the accelerations produced in two bodies by the same force is the inverse ratio of their masses,

i.e. $\frac{a_1}{a_2} = \frac{m_2}{m_1}$ so that $m_1a_1 = m_2a_2 =$ a constant, proportional to the same force.

The standard unit of mass is the kilogram (kg). Mass is a scalar quantity.

The **momentum** p of a body is the product of its mass and velocity: $p = mv$

Because the standard unit of mass is the kilogram (kg) and the standard unit of velocity is the metre per second (m s^{-1}), the standard unit of momentum is the kilogram metre per second (kg m s^{-1}). This is a vector quantity that has the same direction as the velocity.

For example, if a body of mass 5 kg is moving with a velocity of $10\,\text{m s}^{-1}$, its momentum is $50\,\text{kg m s}^{-1}$. According to Newton's first law of motion, this body is unable by itself to change its velocity, and hence is also unable to change its momentum, unless it is acted on by a non-zero resultant force.

This leads to a statement of Newton's second law of motion:

The rate of change of momentum is proportional to the applied force and occurs in the direction of the force.

If a body of mass m is acted on by a non-zero resultant force F, then:

$$F \propto \text{rate of change of momentum}$$

$$\propto \frac{d}{dt}(mv)$$

$$\propto m\frac{dv}{dt} \quad \text{if } m \text{ is constant}$$

$$\therefore \quad F \propto ma \qquad \text{where } a = \frac{dv}{dt}$$

That is, $F = kma$ where k is a constant.

By a suitable selection of units, you can make $k = 1$. If 1 unit of force is defined as the amount of force required to produce an acceleration of $1\,\text{m}\,\text{s}^{-1}$ in a body of mass 1 kg, then: $1 = k \times 1 \times 1 \quad \therefore k = 1$

Hence $F = ma$.

This standard unit of force is called a newton (N) where $1\,\text{N} = 1\,\text{kg}\,\text{m}\,\text{s}^{-2}$.

Alternative definition for Newton's second law of motion

The acceleration of a body is proportional to the resultant force that acts on the body and inversely proportional to the mass of the body.

Look at the formula $F = kma$ from the previous definition.

If k and m are constant, then $F \propto a$ and $a \propto F$, which satisfies the alternative definition.

If F and k are constant, then since ma = constant then $a \propto \frac{1}{m}$, which satisfies the alternative definition.

This definition says that 'The acceleration of a body is proportional to the resultant force that acts on the body', which can be written $a \propto F$, or removing the proportionality that $F = Ka$, where K is a constant.

The m, which is a constant in a given situation, is introduced by taking $K = km$, as this makes later calculations easier.

Newton's third law of motion

When two objects exert force on each other, the forces are equal in magnitude but opposite in direction. In other words: For every action there is an equal but opposite reaction.

Dynamics of a particle

You will consider all bodies as particles, so that all external forces acting on a body are regarded as acting through a single point in the body and producing only a translational effect (i.e. no rotation).

You have seen in section 6.1 that acceleration can take different forms apart from $\frac{dv}{dt}$. For example, using the chain rule:

$$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$$

$$= v\frac{dv}{dx} \qquad \text{as } v = \frac{dx}{dt}$$

$$= \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx}$$

$$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Hence acceleration may be expressed in any of the forms: $\frac{dv}{dt}, \frac{d^2x}{dt^2}, v\frac{dv}{dx}, \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

The form to use in a particular problem will depend on the form of the equation that defines acceleration or force:

- Given $a = f(t)$, use $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$
- Given $a = g(x)$, use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$
- Given $a = h(v)$, use $\begin{cases} \frac{dv}{dt} & \text{if initial conditions are values for } t \text{ and } v \\ v\frac{dv}{dx} & \text{if initial conditions are values for } x \text{ and } v \end{cases}$

Derivatives with respect to time are often written using dots above the dependent variable, e.g. $\dot{x} = \frac{dx}{dt}, \ddot{x} = \frac{d^2x}{dt^2}, \dot{v} = \frac{dv}{dt}$

Note also that for constant m you have $F = m\frac{dv}{dt}$ and so you obtain: $\frac{dv}{dt} = \frac{F}{m}$

The derivative on the left-hand side is the acceleration, which forms the basis for the solution of a differential equation. As m is constant, the problem is converted to a kinematics problem, for which previous methods can be applied.

Some problems may be set in terms of acceleration and some in terms of force.

Example 17

A particle of mass 4 kg is acted on by a force whose direction is constant and whose magnitude at time t seconds is $(12t - 3t^2)$ newtons. If the particle has an initial velocity of 2 m s^{-1} in the direction of the force, find the velocity after 4 seconds.

Solution

There is only one external force acting on the particle, so $F = ma$ becomes:

$$12t - 3t^2 = 4\frac{dv}{dt}$$

where the particle moves with a variable velocity of magnitude $v\text{ m s}^{-1}$ in the direction of the force. Hence:

$$\frac{dv}{dt} = 3t - \frac{3}{4}t^2$$

There are two different methods to complete the solution for the given initial conditions:

Method 1

$$v = \int\left(3t - \frac{3}{4}t^2\right)dt$$
$$= \frac{3}{2}t^2 - \frac{1}{4}t^3 + C$$

When $t = 0$: $v = 2$ and so $C = 2$.

$$\therefore \quad v = \frac{3}{2}t^2 - \frac{1}{4}t^3 + 2$$

When $t = 4$: $v = 10$

Method 2

$$dv = \left(3t - \frac{3}{4}t^2\right)dt$$
$$\int_{\text{initial } v}^{\text{final } v} dv = \int_{\text{initial } t}^{\text{final } t}\left(3t - \frac{3}{4}t^2\right)dt$$
$$\int_2^v dv = \int_0^4\left(3t - \frac{3}{4}t^2\right)dt$$
$$[v]_2^v = \left[\frac{3}{2}t^2 - \frac{1}{4}t^3\right]_0^4$$
$$v - 2 = 24 - 16 - (0 - 0)$$
$$v = 10$$

Hence the velocity after 4 seconds has a magnitude of $10\,\text{m s}^{-1}$.

Example 18

A particle of mass 2 kg moves in a straight line so that at time t seconds its displacement from a fixed origin is x metres and its velocity is $v\,\text{m s}^{-1}$. If the resultant force (in newtons) that acts on the particle is:

(a) $6 - 4x$, find v in terms of x given that $v = 2$ when $x = 1$

(b) $8 - 2v^2$, find t in terms of v given that the particle is initially at rest

(c) $8 - 2v^2$, find x in terms of v given that the particle is initially at the origin.

Solution

(a) $F = m\ddot{x}$: $\quad 2\ddot{x} = 6 - 4x$

Hence the equation of motion is $\ddot{x} = 3 - 2x$. As the question requires v in terms of x and the initial conditions are in v and x, you can use either $\ddot{x} = v\frac{dv}{dx}$ or $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$:

Method 1

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - 2x$$
$$\frac{1}{2}v^2 = \int(3 - 2x)\,dx$$
$$\frac{1}{2}v^2 = 3x - x^2 + C$$

When $v = 2, x = 1$

$\therefore\ 2 = 3 - 1 + C$, so $C = 0$:

$$\frac{1}{2}v^2 = 3x - x^2$$
$$v^2 = 6x - 2x^2$$
$$v = \pm\sqrt{6x - 2x^2}$$

Method 2

$$v\frac{dv}{dx} = 3 - 2x$$
$$v\,dv = (3 - 2x)\,dx$$

From start $v = 2, x = 1$ to end $v = v, x = x$:

$$\int_2^v v\,dv = \int_1^x (3 - 2x)\,dx$$
$$\left[\frac{v^2}{2}\right]_2^v = \left[3x - x^2\right]_1^x$$
$$\frac{v^2}{2} - 2 = 3x - x^2 - (3 - 1)$$
$$\frac{v^2}{2} = 3x - x^2$$
$$v^2 = 6x - 2x^2$$
$$v = \pm\sqrt{6x - 2x^2}$$

Which solution for the velocity is valid—positive or negative?

On a quick inspection, you might say that because the initial condition is $v = 2$ (i.e. positive), then you should take the positive square root. However, you should recognise from the start of the question that the motion is simple harmonic, as: $\ddot{x} = -2\left(x - \frac{3}{2}\right)$

Hence the particle moves both left *and* right, and so both solutions are valid: $v = \pm\sqrt{6x - 2x^2}$

In future examples, Method 2 will be used.

(b) $m\ddot{x} = 8 - 2v^2$ and $m = 2$, so the equation of motion is: $\ddot{x} = 4 - v^2$

You require t in terms of v, hence: $\frac{dv}{dt} = 4 - v^2$

$$\frac{dt}{dv} = \frac{1}{4 - v^2}, \qquad v \neq \pm 2$$

$$\int_0^t dt = \int_0^v \frac{dv}{4 - v^2}$$

Use partial fractions: $\frac{1}{4 - v^2} = \frac{1}{(2 - v)(2 + v)} = \frac{1}{4}\left(\frac{1}{2 - v} + \frac{1}{2 + v}\right)$

$$[t]_0^t = \frac{1}{4}\left[\log_e\left|\frac{2 + v}{2 - v}\right|\right]_0^v$$

$$t = \frac{1}{4}\log_e \frac{2 + v}{2 - v}, \qquad -2 < v < 2$$

(c) $m\ddot{x} = 8 - 2v^2$ and $m = 2$, so the equation of motion is: $\ddot{x} = 4 - v^2$

You require x in terms of v, hence: $v\frac{dv}{dx} = 4 - v^2$

$$\frac{dv}{dx} = \frac{4 - v^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{4 - v^2}$$

$$\int_0^x dx = \int_0^v \frac{v}{4 - v^2}\,dv \quad \text{(from start } x = 0, v = 0 \text{ to end } x = x, v = v\text{)}$$

$$[x]_0^x = -\frac{1}{2}\left[\log_e\left|4 - v^2\right|\right]_0^v$$

$$x = -\frac{1}{2}\left(\log_e\left|4 - v^2\right| - \log_e 4\right)$$

$$x = \frac{1}{2}\log_e\left|\frac{4}{4 - v^2}\right|$$

Example 19

Assume that Earth is a sphere of radius R and that at any point $x \geq R$ distant from the centre of Earth, the acceleration due to gravity is proportional to x^{-2} and is directed towards Earth's centre. Ignore all forces other than Earth's gravity.

A body is projected vertically upwards from the surface of Earth with initial speed V.

(a) Show that the equation of motion of the particle is $\ddot{x} = -\frac{gR^2}{x^2}$, where g is the acceleration due to gravity at Earth's surface.

(b) Show that the velocity v of the particle during its flight is given by: $v^2 = V^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$

(c) Prove that the body's 'escape velocity' is $\sqrt{2gR}$ (i.e. prove that if the particle's initial speed is $V \geq \sqrt{2gR}$, then the particle will escape from Earth and never return).

(d) If $V = \sqrt{2gR}$, prove that the time taken to rise to a height R above Earth's surface is $\frac{1}{3}(4 - \sqrt{2})\sqrt{\frac{R}{g}}$.

Solution

(a) Take O as the centre of Earth and define motion away from O as being in the positive direction.

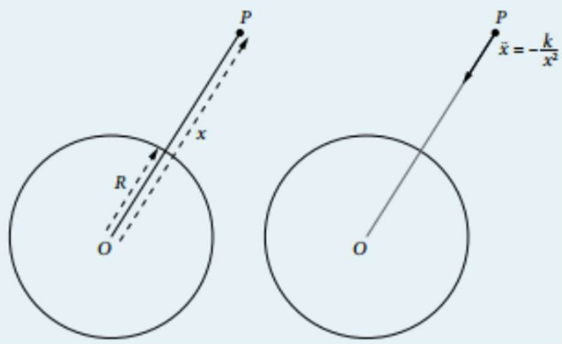
Then: $\ddot{x} = -\frac{k}{x^2}$ [1]

But at $x = R$, $\ddot{x} = -g$

(i.e. at the surface, acceleration due to gravity is g):

$$-g = -\frac{k}{R^2} \quad \therefore k = gR^2$$

Substitute into [1]: $\ddot{x} = -\frac{gR^2}{x^2}$



(b)

$$v\frac{dv}{dx} = -gR^2x^{-2}$$

$$\int_V^v v\,dv = -gR^2\int_R^x x^{-2}\,dx \qquad \text{(from start } v = V, x = R \text{ to end } v = v, x = x\text{)}$$

$$\left[\frac{1}{2}v^2\right]_V^v = gR^2\left[x^{-1}\right]_R^x$$

$$\frac{1}{2}v^2 - \frac{1}{2}V^2 = gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$

$$v^2 = V^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$

(c) If the particle escapes, then $x \to \infty$ and so $\frac{1}{x} \to 0$

From the solution of part **(b)**: $v^2 \to V^2 + 2gR^2\left(0 - \frac{1}{R}\right)$ or $v^2 \to V^2 - 2gR$

But $v^2 \geq 0$ and hence $V^2 - 2gR \geq 0$

i.e. $V^2 \geq 2gR$

As the particle is escaping (i.e. always only moving away from the centre of Earth):

$$V \geq \sqrt{2gR}$$

(d) If $V = \sqrt{2gR}$, the solution of part **(b)** becomes: $v^2 = 2gR + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$

i.e. $v^2 = \frac{2gR^2}{x}$

As you are only concerned with motion away from Earth:

$$v = \sqrt{2g}\,Rx^{-\frac{1}{2}}$$

$$\frac{dx}{dt} = \sqrt{2g}\,Rx^{-\frac{1}{2}}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{2g}\,R}x^{\frac{1}{2}}$$

$$\int_0^t dt = \frac{1}{\sqrt{2g}\,R}\int_R^{2R} x^{\frac{1}{2}}\,dx$$

(Note the limits of the integral on the RHS: from Earth's surface to distance R above Earth's surface.)

$$t = \frac{1}{\sqrt{2g}\,R} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_R^{2R}$$

$$t = \frac{\sqrt{2}}{3R\sqrt{g}}(2R\sqrt{2R} - R\sqrt{R})$$

$$t = \frac{\sqrt{2}}{3R\sqrt{g}}(2\sqrt{2} - 1)R\sqrt{R}$$

$$t = \frac{(4 - \sqrt{2})R\sqrt{R}}{3R\sqrt{g}}$$

$$t = \frac{1}{3}(4 - \sqrt{2})\sqrt{\frac{R}{g}}$$

Example 20

A particle of mass 1 kg is fixed in position, being suspended from the ceiling by two light rods AP and BP. The lengths of AP and BP are in the ratio 3 : 2, and AP is inclined at an angle of 30° below the ceiling. The tension forces in the rods AP and BP are T_1 and T_2 respectively.

(a) Show that the rod BP is inclined at an angle α below the ceiling, where $\sin\alpha = \frac{3}{4}$.

(b) Draw a diagram to show the three forces (T_1, T_2 and the weight force) that act on the particle.

(c) Resolve the forces into horizontal and vertical components.

(d) Use Newton's first law of motion to calculate the values of T_1 and T_2 correct to 1 decimal place. (Use $g = 9.8\,\text{m s}^{-2}$.)

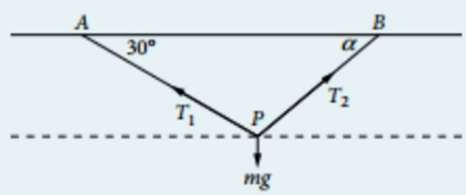
Solution

(a) Using the sine rule in triangle ABP:

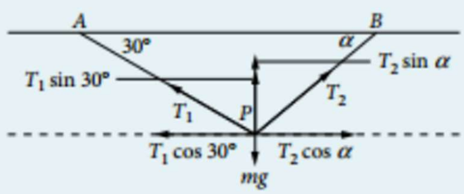
$$\frac{\sin\alpha}{3} = \frac{\sin 30°}{2}$$

$$\sin\alpha = \frac{3}{4}$$

(b)



(c)



(d) The particle is fixed in position, so according to Newton's first law of motion there is a zero resultant force acting horizontally and a zero resultant force acting vertically.

Horizontally: $T_1\cos 30° = T_2\cos\alpha$

$\sin\alpha = \frac{3}{4}$ $\therefore$ $\cos\alpha = \frac{\sqrt{7}}{4}$: $\frac{\sqrt{3}}{2}T_1 = \frac{\sqrt{7}}{4}T_2$ [1]

Vertically (mass $m = 1$): $T_1\sin 30° + T_2\sin\alpha = 9.8$

$\frac{1}{2}T_1 + \frac{3}{4}T_2 = 9.8$ [2]

From [1]: $T_1 = \frac{\sqrt{21}}{6}T_2$

Substitute into [2]: $\frac{\sqrt{21}}{12}T_2 + \frac{3}{4}T_2 = 9.8$

$T_2(\sqrt{21} + 9) = 117.6$

$T_2 = 8.65815\ldots$

Substitute into [1]: $T_1 = 6.61277\ldots$

$\therefore$ Correct to 1 decimal place: $T_1 = 6.6$ and $T_2 = 8.7$