MECHANICS - CHAPTER REVIEW

1 A particle moves on the x-axis with velocity v. The particle is initially at rest at $x = 2$. Its acceleration is given by $\ddot{x} = x + 6$. Using $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, find the speed of the particle at $x = 3$.

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x + 6 \Rightarrow \frac{1}{2}v^2 = \int (x+6)\,dx = \frac{x^2}{2} + 6x + C$

At $t=0$, $x = 2$ and $v = 0$, so $\frac{1}{2} \times 0^2 = \frac{2^2}{2} + 6 \times 2 + C$

so $C = -2 - 12 = -14$

so $\frac{1}{2}v^2 = \frac{x^2}{2} + 6x - 14$ or $v^2 = x^2 + 12x - 28$

At $x = 3$ $v^2 = 3^2 + 12 \times 3 - 28 = 17$

so $v = \sqrt{17}$ at $x = 3$

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2 A particle moves along the x-axis, starting from a position 2 metres to the right of the origin (i.e. $x = 2$ when $t = 0$), with an initial velocity of $\frac{5\sqrt{2}}{2}$ m s^{-1} and an acceleration $\ddot{x} = x^3 + x$.

(a) Show that $\dot{x} = \frac{x^2+1}{\sqrt{2}}$. **(b)** Hence find an expression for x in terms of t.

a) $\ddot{x} = x^3 + x \Rightarrow \frac{d}{dx}\left(\frac{1}{2}v^2\right) = x^3 + x \Rightarrow \frac{1}{2}v^2 = \int (x^3+x)\,dx$

so $\frac{1}{2}v^2 = \frac{x^4}{4} + \frac{x^2}{2} + C$

At $t=0$, $x = 2$ and $v = \frac{5\sqrt{2}}{2}$ so $\frac{1}{2}\left(\frac{5\sqrt{2}}{2}\right)^2 = \frac{2^4}{4} + \frac{2^2}{2} + C$

so $C = \frac{1}{2}\left(\frac{25 \times 2}{4}\right) - 4 - 2 = \frac{1}{4}$ so $\frac{1}{2}v^2 = \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{4} = \frac{1}{4}\left[x^4 + 2x^2 + 1\right]$

so $\frac{1}{2}v^2 = \frac{1}{4}(x^2+1)^2$ so $v^2 = \frac{1}{2}(x^2+1)^2$ so $v = \frac{x^2+1}{\sqrt{2}} = \dot{x}$

b) so $\frac{dx}{dt} = \frac{x^2+1}{\sqrt{2}} \Rightarrow \frac{dx}{1+x^2} = \frac{dt}{\sqrt{2}}$ so $\tan^{-1}x = \frac{1}{\sqrt{2}}t + K$

At $t=0$, $x = 2$, so $\tan^{-1}2 = K$ so $\tan^{-1}x = \frac{1}{\sqrt{2}}t + \tan^{-1}2$

So $x = \tan\left[\frac{t}{\sqrt{2}} + \tan^{-1}2\right] = \frac{\tan(t/\sqrt{2}) + \tan(\tan^{-1}2)}{1 - \tan(t/\sqrt{2})\tan(\tan^{-1}2)}$

$x = \frac{\tan(t/\sqrt{2}) + 2}{1 - 2\tan(t/\sqrt{2})}$

3 The equation of motion for a particle moving in simple harmonic motion is given by $\frac{d^2x}{dt^2} = -n^2x$ where n is a positive constant and x is the displacement of the particle at time t.

(a) Show that the square of the velocity of the particle is $v^2 = n^2(a^2 - x^2)$, where $v = \frac{dx}{dt}$ and a is the amplitude of the motion.

(b) Find the maximum speed of the particle.

(c) Find the maximum acceleration of the particle.

(d) The particle is initially at the origin. Write a formula for x as a function of t. Hence find the first time that the particle's speed is a quarter of its maximum speed.

a) $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2x \Rightarrow \frac{1}{2}v^2 = \int -n^2x\,dx$

So $\frac{1}{2}v^2 = -n^2\frac{x^2}{2} + C$ so $v^2 = -n^2x^2 + 2C = n^2\left(\frac{2C}{n^2} - x^2\right)$

When $v=0$, $x=a$ which is the amplitude of motion.

So $0^2 = n^2\left(\frac{2C}{n^2} - a^2\right)$ so $\frac{2C}{n^2} = a^2$ $\therefore$ $v^2 = n^2(a^2 - x^2)$

b) The speed is maximum when $x=0$, i.e. $v^2_{max} = n^2a^2$ so $v_{max} = na$

c) The acceleration $\frac{d^2x}{dt^2}$ is maximum when $x=-a$; $a_{max} = n^2a$

d) $v^2 = n^2(a^2 - x^2)$ so $v = \frac{dx}{dt} = \pm\sqrt{n^2(a^2-x^2)}$

so $\frac{dx}{dt} = \pm n\sqrt{a^2-x^2}$ so $\frac{dx}{\sqrt{a^2-x^2}} = \pm n\,dt$

So $\int \frac{dx}{\sqrt{a^2-x^2}} = \pm n \int dt$ we do a change of variable $x = a\sin\theta$ so $\frac{dx}{d\theta} = a\cos\theta$

$\Leftrightarrow \int \frac{a\cos\theta\,d\theta}{a\cos\theta} = \pm n \int dt \Leftrightarrow \theta = \pm nt + C$

At $t=0$, $\theta = 0$, so $C=0$ so $\theta = \pm nt$

So $x = a\sin(\pm nt)$ $\dot{x} = a \times (\pm n)\cos(\pm nt) = \pm an\cos(\pm nt)$

$\dot{x} = \frac{1}{4}V_{max} = \frac{na}{4} = \pm an\cos(\pm nt)$ when $\cos(\pm nt) = \frac{1}{4}$ i.e. $t = \frac{1}{n}\cos^{-1}\left(\frac{1}{4}\right)$

when $t = \frac{1}{n}\cos^{-1}(1/4)$, $\dot{x}$ is $1/4$ of the maximum speed.

4 A particle moves with simple harmonic motion on the x-axis about the origin. It is initially at its extreme negative position. The amplitude of the motion is 16 and the particle returns to its initial position every 5 seconds.

(a) Write an equation for the position of the particle at time t seconds.

(b) How much time does the particle take to move from a rest position to the point halfway between the rest position and the equilibrium position?

a) $x = 16\sin(\omega t + \varphi)$

At $t=0$ $x=-16$ so $-16 = 16\sin(\omega\times 0+\varphi) = 16\sin\varphi$

so $\sin\varphi = -1$ so $\varphi = -\pi/2$ $\therefore x = 16\sin\left(\omega t - \frac{\pi}{2}\right)$

For the period, $\omega 5 = 2\pi$, so $5\omega = 2\pi$. so $\omega = \frac{2\pi}{5}$

$\therefore$ $x = 16\sin\left(\frac{2\pi}{5}t - \frac{\pi}{2}\right) \Leftrightarrow x = 16\cos\left(\frac{2\pi t}{5}+\pi\right)$.

b) The rest position is when $x = \pm 16$, i.e. $t=0$ or $t=\frac{k2\pi}{5}$

The equilibrium position is when $x=0$.

So halfway between the rest position and the equilibrium position is for $x=-8$

When $x=-8$, $-8 = 16\sin\left(\frac{2\pi t}{5} - \frac{\pi}{2}\right)$.

$\Leftrightarrow \sin\left(\frac{2\pi t}{5}-\frac{\pi}{2}\right) = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$

so $\frac{2\pi t}{5} - \frac{\pi}{2} = -\frac{\pi}{6} \Leftrightarrow \frac{2t}{5} = \frac{-1}{6}+\frac{1}{2} = \frac{1}{3}$

So $x=-8$ when $t = \frac{5}{6}$ s.

5 A particle moves in a straight line. Its displacement x metres after t seconds is $x = \sin 2t - \sqrt{3}\cos 2t + 3$.

(a) Prove that the particle is moving in simple harmonic motion about $x = 3$ by showing that $\ddot{x} = -4(x-3)$.

(b) What is the period of the motion?

(c) Express the velocity of the particle in the form $\dot{x} = A\cos(2t - \alpha)$, where α is in radians.

(d) Hence, or otherwise, find all times within the first π seconds when the particle is moving at 2 metres per second in either direction.

a) $\dot{x} = 2\cos 2t - 2\sqrt{3}(-\sin 2t) = 2\cos 2t + 2\sqrt{3}\sin 2t$

$\ddot{x} = 4(-\sin 2t) + 4\sqrt{3}\cos 2t = -4(\sin 2t - \sqrt{3}\cos 2t)$

So $\ddot{x} = -4(x-3)$

The relation is similar to $\ddot{x} = -n^2 x$, so it's SHM.

b) Period is $2T = 2\pi$, so $T = \pi$ s.

c) $\dot{x} = 2\cos 2t + 2\sqrt{3}\sin 2t = 2\left[\cos 2t + \sqrt{3}\sin 2t\right]$

so $\dot{x} = 4\left[\frac{1}{2}\cos 2t + \frac{\sqrt{3}}{2}\sin 2t\right] = 4\left[\cos\frac{\pi}{3}\cos 2t + \sin\frac{\pi}{3}\sin 2t\right]$

so $\dot{x} = 4\cos\left(\frac{\pi}{3} - 2t\right) = 4\cos\left(2t - \frac{\pi}{3}\right)$

d) $\dot{x} = \pm 2$ when $\pm 2 = 4\cos\left(2t - \frac{\pi}{3}\right) \Leftrightarrow \cos\left(2t - \frac{\pi}{3}\right) = \pm\frac{1}{2}$

<u>Case 1</u> $\cos\left(2t - \frac{\pi}{3}\right) = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) \Rightarrow 2t - \frac{\pi}{3} = \pm\frac{\pi}{3} + 2n\pi$ (n integer)

so $2t = \pm\frac{\pi}{3} + \frac{\pi}{3} + 2n\pi \Leftrightarrow t = \pm\frac{\pi}{6} + \frac{\pi}{6} + n\pi$

$n = 0$ gives $t = 0$ or $t = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$

$n = 1$ gives $t = \pi$ or $t = \pi + \frac{\pi}{3}$ ← this one outside $[0, \pi]$

<u>Case 2</u> $\cos\left(2t - \frac{\pi}{3}\right) = -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right) \Rightarrow 2t - \frac{\pi}{3} = \pm\frac{2\pi}{3} + 2n\pi$ (n integer)

so $2t = \pm\frac{2\pi}{3} + \frac{\pi}{3} + 2n\pi \Leftrightarrow t = \pm\frac{\pi}{3} + \frac{\pi}{6} + n\pi$

$n = 0$ gives $t = \pi/2$ $n = 1$ gives $t = \frac{5\pi}{6}$

$n = 2$ gives $t = \frac{11\pi}{6}$ <u>outside</u>

So: $t = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

6 A particle is moving along the x-axis and is initially at the origin. Its velocity v metres per second at time t seconds is given by $v = \frac{2t}{9+t^2}$.

(a) What is the initial velocity of the particle?
(b) Find an expression for the acceleration of the particle. **(c)** When is the acceleration zero?
(d) What is the maximum velocity attained by the particle and when does it occur?
(e) Find the position of the particle when $t = 3$.

a) At $t=0$ $\quad v = \frac{2\times 0}{9+0^2} = 0$

b) $a = \frac{dv}{dt} = \frac{d}{dt}\left[\frac{2t}{9+t^2}\right] = \frac{2(9+t^2) - 2t\times 2t}{(9+t^2)^2} = \frac{-2t^2+18}{(9+t^2)^2}$

c) $a = 0$ when $-2t^2+18 = 0 \iff t^2 = 9$ so $t = 3s$

d) The velocity is at its maximum when $a=0$, i.e. when $t=3$

At $t=3$, $\quad v = \frac{2\times 3}{9+3^2} = \frac{6}{18} = \frac{1}{3}\ ms^{-1}$

e) $v = \frac{dx}{dt}$ so $\frac{dx}{dt} = \frac{2t}{9+t^2}$

so $dx = \frac{2t\,dt}{9+t^2}$ so $x = \int \frac{2t\,dt}{9+t^2}$

$x = \ln|9+t^2| + C$

At $t=0$, the particle is said to be at the origin, so $x=0$

so $0 = \ln|9+0^2| + C$ so $C = -\ln 9$

So $x = \ln\left|\frac{9+t^2}{9}\right| = \ln\left|1+\left(\frac{t}{3}\right)^2\right|$

At $t=3$, $\quad x(3) = \ln\left|1+\left(\frac{3}{3}\right)^2\right| = \ln 2$

7 A particle of mass 5 kg moves in a straight line under the action of a force whose magnitude after t seconds is $50 - 10t$ N. Initially the particle is at the origin O with velocity $24\,\mathrm{m\,s^{-1}}$.

(a) At what time is the particle momentarily at rest? **(b)** What is its position at that time?

(c) Describe the motion.

a) $F = ma$

So $50 - 10t = m\ddot{x}$ so $\ddot{x} = \dfrac{50 - 10t}{5} = 10 - 2t$

$\therefore \dot{x} = 10t - t^2 + C$

At $t = 0$, $\dot{x}(0) = 24$ so $C = 24$ $\therefore \dot{x}(t) = -t^2 + 10t + 24$

$\dot{x}(t) = 0$ when $-t^2 + 10t + 24 = 0$ $\Delta = 100 + 4 \times 24 = 196 = 14^2$

So $\dot{x}(t) = 0$ when $t = \dfrac{-10 \pm 14}{(-2)}$ $t = 12\,s$ or $t = -2$ (impossible)

So the particle is momentarily at rest when $t = 12\,s$.

b) $x(t) = 10\dfrac{t^2}{2} - \dfrac{t^3}{3} + 24t + K$

$K = 0$ as the particle is at the origin at $t = 0$.

So $x(t) = -\dfrac{t^3}{3} + 5t^2 + 24t$

$x(12) = -\dfrac{12^3}{3} + 5 \times 12^2 + 24 \times 12 = 432\,m$

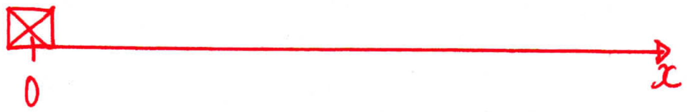
c) Particle moves right (i.e. in the positive direction) from O, slowing until momentarily at rest after 12 s, then moves left with increasing speed.

8 An object of mass 10 kg is at rest at the origin. It is acted on by a force that decreases uniformly with the distance travelled by the object, from 50 N at the start to 10 N when the distance travelled is 25 m.

(a) Write the function for this force F in terms of displacement x.

(b) Find the velocity of the object when its displacement is 25 m.

a) $F = kx + F(0)$



At $x = 0$ $\quad F = F(0) = 50$

At $x = 25$ $\quad F = k \times 25 + 50 = 10$ $\quad$ so $k = \dfrac{-40}{25} = -1.6$

$F = -1.6x + 50$

b) $F = m\ddot{x}$ $\quad$ so $\ddot{x} = \dfrac{F}{m} = \dfrac{-1.6x + 50}{10}$

$\ddot{x} = -0.16x + 5$

But $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ $\quad$ so $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = -0.16x + 5$

So $\dfrac{1}{2}v^2 = \int(-0.16x + 5)\,dx = -0.16\dfrac{x^2}{2} + 5x + C$

At $x = 0$, $v = 0$ $\quad$ so $C = 0$.

$v^2 = -0.16x^2 + 10x$

At $x = 25\,m$ $\quad v^2 = 10 \times 25 - 0.16 \times 25^2 = 150$

So $v = \sqrt{150} = \sqrt{6 \times 5^2} = 5\sqrt{6}$ ms^{-1}

9 A particle of mass 5 kg moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v.

(a) If the resultant force (in newtons) on the particle is $10\sin t$, and $v = 1$ and $x = 1$ when $t = 0$, then find x as a function of t.

(b) If the resultant force (in newtons) on the particle is $15 + 5v$, and $v = 0$ when $t = 0$, then find v as a function of t.

a) $F = 10\sin t = m\ddot{x} = m\dfrac{d^2x}{dt^2}$

$\ddot{x} = \dfrac{10}{m}\sin t = \dfrac{10}{5}\sin t = 2\sin t.$

$\dot{x} = \int 2\sin t \, dt = 2(-\cos t) + C$

At $t = 0$, $\dot{x}(0) = 1$ so $1 = -2\cos 0 + C = -2 + C$

So $C = 3$.

$\dot{x} = -2\cos t + 3$

$x = \int(-2\cos t + 3)\,dt = -2\sin t + 3t + K$

At $t = 0$ $x(0) = 1$ so $1 = -2\sin 0 + 3\times 0 + K$

so $K = 1$

$x = -2\sin t + 3t + 1$

b) $\ddot{x} = \dfrac{F}{m} = \dfrac{15 + 5v}{5} = 3 + v.$

So $\dfrac{dv}{dt} = 3 + v \Rightarrow \dfrac{dv}{3+v} = dt$

$\int \dfrac{dv}{3+v} = \int dt$ so $\ln|3+v| = t + C$

so $3 + v = e^{t+C} = Ae^{t}$ so $v = Ae^{t} - 3$

At $t = 0$, $v = 0$, so $0 = Ae^{0} - 3 = A - 3$ $\therefore A = 3$

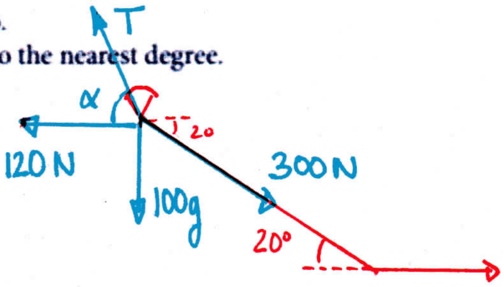
$v = 3e^{t} - 3$

10 A parasailing waterskier is being towed horizontally at a constant speed. The tow rope from the boat makes an angle of 20° above the horizontal and there is tension of 300 N in the tow rope. The waterskier has a mass of 100 kg. A resistance force of 120 N acts against the waterskier in a horizontal direction. A parachute is attached to the skier by a cord that is inclined at an angle α above the horizontal. There is tension of T newtons in the parachute cord. (Use $g = 9.8\ \mathrm{m\,s^{-2}}$.)

(a) Draw a diagram to show the four forces acting on the waterskier, W.

(b) Explain why the resultant force on the waterskier is zero.

(c) Find T correct to one decimal place and find α correct to the nearest degree.



b) The waterskier is moving with constant speed in a straight line. By Newton's first law of motion, either there is no force or the resultant of all forces is zero.

c) We sum all the forces on the horizontal and vertical.

Horizontal $\quad 300\cos 20 = 120 + T\cos\alpha \quad ①$

Vertical $\quad 100g + 300\sin 20 = T\sin\alpha \quad ②$

$① \Rightarrow \quad T\cos\alpha = 300\cos 20 - 120$

so $T^2\cos^2\alpha = [300\cos 20 - 120]^2 \quad Ⓐ$

$② \Rightarrow \quad T\sin\alpha = 100g + 300\sin 20$

so $T^2\sin^2\alpha = [100g + 300\sin 20]^2 \quad Ⓑ$

Adding Ⓐ and Ⓑ gives: $T^2 = [300\cos 20 - 120]^2 + [100g + 300\sin 20]^2$

$T \simeq 1{,}094.6\,\mathrm{N}$ with $g = 9.8$

So $\sin\alpha = \dfrac{980 + 300\sin 20}{1{,}094.6} \Rightarrow \alpha \simeq 82°$

11 An object is fired vertically from the surface of the Moon with initial velocity v_0 under a gravitational acceleration such that $\ddot{x} = -\frac{k}{x^2}$, where x is the displacement from the centre of the Moon and k is a constant. Let the radius of the Moon be R. The gravitational acceleration at the surface of the Moon is $\frac{g}{6}$.

(a) Find the velocity of the object in terms of its distance x from the centre of the Moon.

(b) Find the value of v_0 for which the object travels a distance of $2R$ from launch before it starts to fall back.

(c) Find the escape velocity.

a) $\ddot{x} = -\frac{k}{x^2}$

When $x = R$ $\quad \ddot{x} = \frac{-g}{6}$ $\quad$ so $-\frac{k}{R^2} = \frac{-g}{6}$ $\quad$ so $k = +\frac{gR^2}{6}$

So $\ddot{x} = \frac{-gR^2}{6x^2}$ $\quad$ But $\ddot{x} = v\frac{dv}{dx}$ $\quad$ so $v\frac{dv}{dx} = -\frac{gR^2}{6x^2}$

$\therefore$ $v\,dv = -\frac{gR^2}{6}\frac{dx}{x^2}$ $\quad \Rightarrow \int v\,dv = \int -\frac{gR^2}{6}\frac{dx}{x^2}$

$\therefore$ $\frac{v^2}{2} = -\frac{gR^2}{6}\left[\frac{x^{-2+1}}{-2+1}\right] + C$ $\quad$ so $v^2 = \frac{gR^2}{3} \times \frac{1}{x} + C$

When $x = R$ $\quad v = v_0$ $\quad$ so $v_0^2 = \frac{gR^2}{3R} + C$ $\quad$ so $C = v_0^2 - \frac{gR}{3}$

So $v^2 - v_0^2 = \frac{gR^2}{3}\left[\frac{1}{x} - \frac{1}{R}\right]$

b) $v = 0$ when $x = 2R$, i.e. $-v_0^2 = \frac{gR^2}{3}\left[\frac{1}{2R} - \frac{1}{R}\right]$

so $v_0^2 = \frac{gR^2}{3}\left[\frac{1}{R} - \frac{1}{2R}\right] = \frac{gR^2}{3 \times 2R} = \frac{gR}{6}$

so $v_0 = \sqrt{\frac{gR}{6}}$

c) The escape velocity is when $x \to +\infty$, i.e. $1/x \to 0$.

So $v^2 - v_0^2 = \frac{gR^2}{3}\left[\frac{1}{x} - \frac{1}{R}\right]$ becomes $v^2 - v_0^2 = \frac{gR^2}{3}\left[0 - \frac{1}{R}\right]$

$\Leftrightarrow$ $v_0^2 - v^2 = \frac{gR}{3}$ $\quad$ The minimum value of v_0 is when $v = 0$

For $v = 0$, $\quad v_0^2 = \frac{gR}{3}$ $\quad$ OR $\quad v_0 = \sqrt{\frac{gR}{3}}$

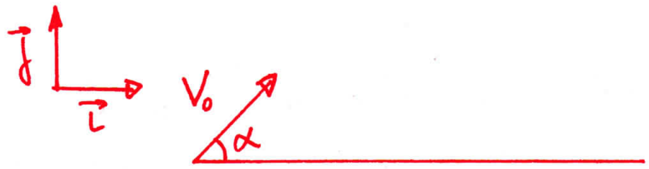
13 **(a)** Show that the range on a horizontal plane of a particle projected upwards at an angle α to the plane and with velocity V metres per second is $\frac{V^2 \sin 2\alpha}{g}$ metres, and that the maximum range is $\frac{V^2}{g}$.

A garden sprinkler sprays water symmetrically about its vertical axis at a constant speed of V metres per second in a circular pattern. The direction of the spray varies continuously between angles of 15° and 60° to the horizontal.

(b) Prove that the sprinkler, from a fixed position on level ground, will wet the surface of an annular region with centre O and with internal and external radii $\frac{V^2}{2g}$ metres and $\frac{V^2}{g}$ metres respectively.

(c) Deduce that if the sprinkler is placed appropriately relative to a rectangular garden bed of size 6 m by 3 m, then the entire garden bed may be watered, provided that $\frac{V^2}{2g} \geq 1 + \sqrt{7}$.

a) $\vec{a} = -g\vec{j}$



So $\vec{V} = -gt\vec{j} + \vec{C}$

At $t=0$ $\quad \vec{V} = V_0 \cos\alpha\,\vec{i} + V_0 \sin\alpha\,\vec{j}$

so $\vec{V} = -gt\vec{j} + V_0\cos\alpha\,\vec{i} + V_0 \sin\alpha\,\vec{j}$

$\vec{V} = [V_0\cos\alpha]\vec{i} + [V_0\sin\alpha - gt]\vec{j} = \frac{d\vec{r}}{dt}$

so $\vec{r} = [V_0\cos\alpha\, t]\vec{i} + [V_0 \sin\alpha\, t - \frac{1}{2}gt^2]\vec{j} + \vec{K}$

At $t=0$, $\vec{r}(0) = \vec{0}$, so $\vec{K} = \vec{0}$, and so:

$$\begin{cases} x(t) = V_0\cos\alpha\, t \\ y(t) = V_0\sin\alpha\, t - \frac{1}{2}gt^2 \end{cases} \iff \begin{cases} t = x/V_0\cos\alpha \\ y = \frac{\cancel{V_0}\sin\alpha \times x}{\cancel{V_0}\cos\alpha} - \frac{1}{2}g\left[\frac{x}{V_0\cos\alpha}\right]^2 \end{cases}$$

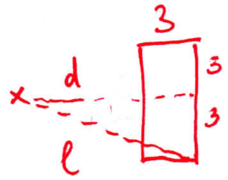
$y = \tan\alpha\, x - \frac{gx^2}{2V_0^2\cos^2\alpha} = x\left[\tan\alpha - \frac{gx}{2V_0^2\cos^2\alpha}\right]$

$y = 0$ when $x = 0$ OR $\tan\alpha = \frac{gx}{2V_0^2\cos^2\alpha}$, i.e. $x = \frac{2V_0^2\cos^2\alpha}{g}\frac{\sin\alpha}{\cos\alpha}$

i.e. $x = \frac{V_0^2 \sin 2\alpha}{g}$ which is maximum when $\alpha = 45$, and is $x = \frac{V_0^2}{g}$.

b) When $\alpha = 15°$ Range is $\frac{V_0^2 \sin 30}{g} = \frac{V_0^2}{2g}$ So indeed between $\frac{V_0^2}{2g}$ and $\frac{V_0^2}{g}$.

c)



d for 15° Range is $V_0^2/2g$

l for 45° $l = \sqrt{3^2 + (d+3)^2}$ Range is V_0^2/g.

So we must have $(2d)^2 = 3^2 + (d+3)^2$

$\iff 3d^2 - 6d - 18 = 0 \iff d^2 - 2d - 6 = 0$

$d = 1 \pm \sqrt{7}$ But $d > 0$

so $d = 1 + \sqrt{7}$ so $\frac{V^2}{2g} > 1 + \sqrt{7}$

14 An underwater camera of mass 0.5 kg is allowed to fall vertically from the ocean surface into a deep ocean trench. As it falls to the ocean floor, it is acted upon by gravity and by a resistance of $2v$ newtons, where $v\ \mathrm{m\,s^{-1}}$ is the velocity of the camera t seconds after beginning its descent.

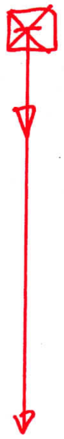
(a) Show that the equation of motion of the camera is $\ddot{x} = g - 4v$.

(b) Find v as a function of t.

(c) Find the terminal velocity of the camera.

(d) Find the time taken for the camera to reach half of its terminal velocity.

(e) It takes 50 seconds for the camera to reach the ocean floor. Find the depth of the ocean at that point.



a) $m\ddot{x} = mg - 2v$ so $\ddot{x} = g - \frac{2v}{0.5} = g - 4v$

b) $\ddot{x} = \frac{dv}{dt}$ so $\frac{dv}{dt} = g - 4v \Leftrightarrow \frac{dv}{g-4v} = dt$

$\Rightarrow -\frac{1}{4}\ln|g-4v| = t + C \Rightarrow g - 4v = Ae^{-t}$

so $v = \frac{1}{4}\left(g - Ae^{-4t}\right)$ At $t = 0$, $v = 0$ so $0 = \frac{1}{4}(g - A)$ so $A = g$.

$v = \frac{g}{4}\left(1 - e^{-4t}\right)$

c) Terminal velocity is when $t \to +\infty$
this is $v = g/4$

d) half of the terminal velocity is $g/8$.

$g/8 = \frac{g}{4}\left(1 - e^{-4t_h}\right) \Rightarrow 1 - e^{-4t_h} = \frac{1}{2} \Rightarrow e^{-4t_h} = \frac{1}{2}$

So $-4t_h = \ln\frac{1}{2} = -\ln 2$ so $t_h = \frac{1}{4}\ln 2 \simeq 0.17$ s.

e) $v = \frac{dx}{dt}$ so $\frac{dx}{dt} = \frac{g}{4}\left(1 - e^{-4t}\right)$ so $dx = \frac{g}{4}\left(1 - e^{-4t}\right)dt$

$\int_0^h dx = \frac{g}{4}\int_0^T \left(1 - e^{-4t}\right)dt = \frac{g}{4}\left[t + \frac{1}{4}e^{-4t}\right]_0^T = \frac{g}{4}\left[T + \frac{1}{4}e^{-4T} - \frac{1}{4}\right]$

So $h = \frac{g}{4}\left[T + \frac{1}{4}e^{-4T} - \frac{1}{4}\right]$

For $T = 50$ $h = \frac{g}{4}\left[50 + \underbrace{\frac{1}{4}e^{-4\times 50}}_{\simeq 0} - \frac{1}{4}\right] \simeq \frac{199g}{16} \simeq 31$ seconds.

15 A particle moves so that its position vector $\underset{\sim}{r}$ at time t is given by $\underset{\sim}{r} = 3\cos 2t\,\underset{\sim}{i} + 3\sin 2t\,\underset{\sim}{j}$, $t \geq 0$.

(a) Show that the particle moves in a circle and find the Cartesian equation of its path.

(b) Show that the particle moves with constant speed.

(c) Show that the particle's acceleration has constant magnitude and is perpendicular to the direction of motion of the particle.

a) $\begin{cases} x(t) = 3\cos 2t \\ y(t) = 3\sin 2t \end{cases}$ so $\begin{cases} x^2 = 9\cos^2 2t \\ y^2 = 9\sin^2 2t \end{cases}$ so $x^2 + y^2 = 9$

circle centre O, radius 3

b) $\frac{d\vec{r}}{dt} = 3 \times 2 \times (-\sin 2t)\vec{i} + 3 \times 2 \times \cos 2t\,\vec{j}$

$\vec{v} = \frac{d\vec{r}}{dt} = -6\sin 2t\,\vec{i} + 6\cos 2t\,\vec{j}$

$|\vec{v}| = \sqrt{(-6\sin 2t)^2 + (6\cos 2t)^2} = \sqrt{6^2\cos^2 2t + 6^2\sin^2 2t} = 6$

So the particle moves with constant speed, $6\,ms^{-1}$

c) $\vec{a} = \frac{d\vec{v}}{dt} = -6 \times 2\cos 2t\,\vec{i} + 6 \times 2 \times (-\sin 2t)\,\vec{j}$

$\vec{a} = -12\cos 2t\,\vec{i} - 12\sin 2t\,\vec{j}$

So $\vec{a} = -4\vec{r}$

$|\vec{a}| = \sqrt{(-12\cos 2t)^2 + (-12\sin 2t)^2} = \sqrt{12^2} = 12$ which is constant

$\vec{a}\cdot\vec{v} = (-12\cos 2t) \times (-6\sin 2t) + (-12\sin 2t) \times (6\cos 2t)$

$\vec{a}\cdot\vec{v} = 72\cos 2t\sin 2t - 72\sin 2t\cos 2t$

So $\vec{a}\cdot\vec{v} = 0$ so the vectors are perpendicular as their dot product is zero, i.e. the particle's acceleration is prependicular to the direction of motion of the particle.

16 The position vector of a particle at time t seconds, $t \geq 0$, is $\underset{\sim}{r} = (1 + \sin 4t)\underset{\sim}{i} + (2 - \cos 4t)\underset{\sim}{j}$ metres.

(a) Show that the particle moves in a circle and sketch its path.

(b) Show that the particle's acceleration is always perpendicular to its velocity.

a) $\begin{cases} x(t) = 1 + \sin 4t \\ y(t) = 2 - \cos 4t \end{cases} \Leftrightarrow \begin{cases} \sin 4t = x - 1 \\ \cos 4t = 2 - y \end{cases}$

So $\sin^2 4t + \cos^2 4t = (x-1)^2 + (2-y)^2 = 1$

Circle centre $(1,2)$ radius 1

b) $\vec{v} = \dfrac{d\vec{r}}{dt} = 4\cos 4t\, \vec{i} + 4 \sin 4t\, \vec{j}$

$\vec{a} = \dfrac{d\vec{v}}{dt} = 16(-\sin 4t)\, \vec{i} + 16 \cos 4t\, \vec{j}$

So $\vec{a} \cdot \vec{v} = 16 \times 4 (-\sin 4t \cos 4t) + 16 \times 4 \sin 4t \cos 4t$

So $\vec{a} \cdot \vec{v} = -64 \sin 4t \cos 4t + 64 \sin 4t \cos 4t$

So $\vec{a} \cdot \vec{v} = 0$ i.e. vectors are perpendiculars.

The particle's acceleration is always perpendicular to its velocity.

17 The position vector of a particle at time t, $t \geq 0$, is $\underset{\sim}{r} = 2\cos 3t\,\underset{\sim}{i} + 2\sin 3t\,\underset{\sim}{j} + 3t\,\underset{\sim}{k}$.

Show that the magnitudes of the particle's velocity and its acceleration are constant.

$$\vec{V} = \frac{d\vec{r}}{dt} = 2 \times 3\,(-\sin 3t)\,\vec{i} + 2 \times 3 \cos 3t\,\vec{j} + 3\vec{k}$$

So $\vec{V} = -6\sin 3t\,\vec{i} + 6\cos 3t\,\vec{j} + 3\vec{k}$

$$\vec{a} = \frac{d\vec{V}}{dt} = -6 \times 3 \cos 3t\,\vec{i} + 6 \times 3\,(-\sin 3t)\,\vec{j}$$

So $\vec{a} = -18\cos 3t\,\vec{i} - 18\sin 3t\,\vec{j}$

$$|\vec{V}| = \sqrt{(-6\sin 3t)^2 + (6\cos 3t)^2 + 3^2} = \sqrt{6^2 + 3^2} = \sqrt{45}$$

which is a constant

$$|\vec{a}| = \sqrt{(-18\cos 3t)^2 + (-18\sin 3t)^2}$$

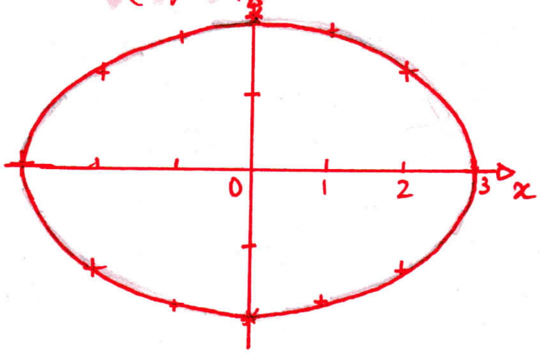
$|\vec{a}| = \sqrt{18^2} = 18$ which is also a constant indeed

MECHANICS - CHAPTER REVIEW

18 A particle moves so that its position vector at time t is given by $\underset{\sim}{r} = 3\cos t\,\underset{\sim}{i} + 2\sin t\,\underset{\sim}{j}$, $0 \le t \le 2\pi$.

(a) Find the Cartesian equation of the path of the particle and sketch the path.
(b) Find when the velocity of the particle is perpendicular to its position vector and hence find the position vectors at these times.
(c) Sketch the graph of the speed function and find the maximum and minimum speeds of the particle.
(d) Show that the particle's acceleration is directed towards the origin and is equal in magnitude to the particle's distance from the origin.
(e) Find when the acceleration is perpendicular to the velocity.

a) $\begin{cases} x(t) = 3\cos t \\ y(t) = 2\sin t \end{cases} \Rightarrow \begin{cases} \cos t = x/3 \\ \sin t = y/2 \end{cases}$ so $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$



b) $\vec{v} = \frac{d\vec{r}}{dt} = -3\sin t\,\vec{i} + 2\cos t\,\vec{j}$

So $\vec{v}\cdot\vec{r} = (-3\sin t)(3\cos t) + (2\cos t)(2\sin t)$

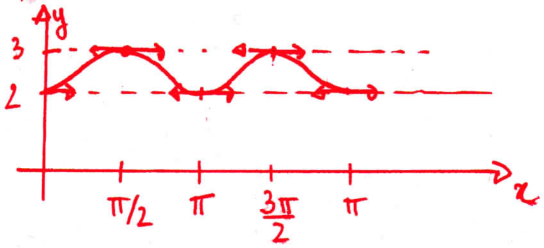
$\vec{v}\cdot\vec{r} = -9\sin t\cos t + 4\sin t\cos t$

$\vec{v}\cdot\vec{r} = -5\sin t\cos t = -\frac{5}{2}\sin 2t$

So $\vec{v}\cdot\vec{r} = 0$ when $\sin 2t = 0$ or $2t = (-1)^n \times 0 + n\pi$

So when $t = n\frac{\pi}{2}$, i.e. at $t=0$, $t=\frac{\pi}{2}$, $t=\pi$ and $t=\frac{3\pi}{2}$

c) $|\vec{v}| = \sqrt{(-3\sin t)^2 + (2\cos t)^2} = \sqrt{9\sin^2 t + 4\cos^2 t}$



d) $\vec{a} = \frac{d\vec{v}}{dt} = -3\cos t\,\vec{i} - 2\sin t\,\vec{j}$

So $|\vec{a}| = |\vec{r}|$ and $\vec{a} = -\vec{r}$

$\therefore$ The particle's acceleration is directed towards the origin and is equal in magnitude to the particle's distance from the origin.

e) $\vec{a}\cdot\vec{v} = -\vec{r}\cdot\vec{v} = \frac{5}{2}\sin 2t$ (as demonstrated at b))

So $\vec{a}\cdot\vec{v} = 0$ when $\sin 2t = 0$, i.e. at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$