2 Find: (a) $\int \frac{\sin x}{2-\cos x}\,dx$ (b) $\int \frac{dx}{3+2\cos x}$ (c) $\int \frac{dx}{1+\sin x}$

a) $\int \frac{\sin x}{2-\cos x}\,dx = \int \frac{(2-\cos x)'}{2-\cos x}\,dx = \ln|2-\cos x| + C$

b) $\int \frac{dx}{3+2\cos x} = \int \frac{1}{3+2\times\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{(1+t^2)}\,dt = \int \frac{2\,dt}{3(1+t^2)+2(1-t^2)}$

$\_\_\_ = \int \frac{2\,dt}{t^2+5} = 2\int \frac{1}{t^2+(\sqrt{5})^2}\,dt$

$\_\_\_ = \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{t}{\sqrt{5}}\right) + C = \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{5}}\right) + C$

c) $\int \frac{dx}{1+\sin x} = \int \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$

$\_\_\_ = \int \frac{2}{t^2+2t+1}\,dt = \int \frac{2}{(1+t)^2}\,dt = 2\int \frac{dt}{(1+t)^2}$

Change of variable $X = 1+t \quad dX = dt$

$\_\_\_ = 2\int \frac{dX}{X^2} = 2\int X^{-2}\,dX = 2\,\frac{X^{-2+1}}{(-2+1)} + C = \frac{2X^{-1}}{(-1)} + C$

$\_\_\_ = -\frac{2}{X} + C = -\frac{2}{1+t} + C$

$\_\_\_ = -\frac{2}{1+\tan\left(\frac{x}{2}\right)} + C$

THE SUBSTITUTION t = tan(A/2)

3 Evaluate: **(a)** $\int_0^{\frac{\pi}{3}} \sec x \, dx$ **(b)** $\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{2+\cos\theta} d\theta$

a) $\int \sec x \, dx = \int \frac{1}{\cos x} dx = \int \frac{1}{\frac{1-t^2}{1+t^2}} \times \frac{2\,dt}{1+t^2} = \int \frac{2}{1-t^2} dt$

$\_\_\_ = \int \frac{2}{(1-t)(1+t)} dt = \int \frac{1}{1-t} + \frac{1}{1+t} dt$

$\_\_\_ = -\ln(1-t) + \ln(1+t) + C$

$\_\_\_ = \ln\left(\frac{1+t}{1-t}\right) + C = \ln\left(\frac{1+\tan(x/2)}{1-\tan(x/2)}\right) + C$

So $\int_0^{\pi/3} \sec x \, dx = \left[\ln\left(\frac{1+\tan(x/2)}{1-\tan(x/2)}\right)\right]_0^{\pi/3} = \ln\left(\frac{1+1/\sqrt{3}}{1-1/\sqrt{3}}\right) - \ln\left(\frac{1+0}{1-0}\right)$

$\_\_\_ = \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) = \ln\left(\frac{(\sqrt{3}+1)^2}{3-1}\right) = \ln\left(\frac{3+2\sqrt{3}+1}{2}\right)$

$\_\_\_ = \ln\left(\frac{4+2\sqrt{3}}{2}\right) = \ln(2+\sqrt{3})$

b) $\int_0^{\pi/2} \frac{\sin\theta}{2+\cos\theta} d\theta = \int_0^{\pi/2} \frac{(-\cos\theta)'}{2+\cos\theta} d\theta = -\int_0^{\pi/2} \frac{(2+\cos\theta)'}{2+\cos\theta} d\theta$

$\_\_\_ = -\left[\ln(2+\cos\theta)\right]_0^{\pi/2} = -\left[\ln(2+0) - \ln(2+1)\right] = \ln\left(\frac{3}{2}\right)$

4 Find: (a) $\int \frac{dx}{5+4\cos x}$ $\qquad \cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}\,dt$

$$\int \frac{dx}{5+4\cos x} = \int \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{(1+t^2)}\,dt$$

$$\text{———} = \int \frac{2}{5(1+t^2)+4(1-t^2)}\,dt$$

$$\text{———} = \int \frac{2\,dt}{t^2+9}$$

$$\text{———} = \int \frac{2}{t^2+3^2}\,dt$$

$$\text{———} = \frac{2}{3}\int \frac{3}{t^2+3^2}\,dt$$

$$\text{———} = \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$$

$$\therefore \int \frac{dx}{5+4\cos x} = \frac{2}{3}\tan^{-1}\left(\frac{\tan(x/2)}{3}\right) + C$$

4 Find: (d) $\int \frac{dx}{3-\cos x}$ $\quad \cos x = \frac{1-t^2}{1+t^2} \quad dx = \frac{2}{1+t^2}\,dt$

$$\int \frac{dx}{3-\cos x} = \int \frac{1}{3-\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{(1+t^2)}\,dt$$

$$\text{———} = \int \frac{2}{3(1+t^2)-(1-t^2)}\,dt$$

$$\text{———} = \int \frac{2}{4t^2+2}\,dt = \int \frac{1}{2t^2+1}\,dt$$

$$\text{———} = \frac{1}{2}\int \frac{1}{t^2+\left(\frac{1}{\sqrt{2}}\right)^2}\,dt$$

$$\text{———} = \frac{1}{2}\left[\int \frac{1/\sqrt{2}}{t^2+\left(\frac{1}{\sqrt{2}}\right)^2}\,dt\right] \times \sqrt{2}$$

$$\text{———} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{2}\,t\right) + C$$

$$\text{———} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{2}\tan\left(\frac{x}{2}\right)\right) + C$$

4 Find: (c) $\int \frac{\sin\theta}{2+\sin\theta} d\theta$

Tip: Start by adding and subtracting 2 at the numerator.

$$\int \frac{\sin\theta}{2+\sin\theta} d\theta = \int \frac{\sin\theta + 2 - 2}{2+\sin\theta} d\theta = \int \left[\frac{2+\sin\theta}{2+\sin\theta} - \frac{2}{2+\sin\theta}\right] d\theta$$

$$— = \int \left[1 - \frac{2}{2+\sin\theta}\right] d\theta = \theta - \int \frac{2}{2+\sin\theta} d\theta + C$$

$$— = \theta - 2\int \frac{1}{2 + \frac{2t}{1+t^2}} \times \frac{2}{1+t^2} dt + C$$

$$\sin\theta = \frac{2t}{1+t^2} \qquad d\theta = \frac{2}{1+t^2} dt.$$

$$— = \theta - 2\int \frac{2}{2(1+t^2)+2t} dt + C$$

$$— = \theta - 2\int \frac{dt}{t^2+t+1} + C$$

Now $t^2+t+1 = \left(t+\frac{1}{2}\right)^2 - \frac{1}{4} + 1 = \left(t+\frac{1}{2}\right)^2 + \frac{3}{4} = \left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$

$$— = \theta - 2\int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + C \qquad \text{let } T = t + \frac{1}{2} \qquad dT = dt$$

$$— = \theta - 2\int \frac{dT}{T^2 + (\sqrt{3}/2)^2} + C = \theta - 2\left[\int \frac{(\sqrt{3}/2)\, d\theta}{T^2 + (\sqrt{3}/2)^2}\right] \times \frac{2}{\sqrt{3}} + C$$

$$— = \theta - \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{T}{\frac{\sqrt{3}}{2}}\right) + C = \theta - \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{2T}{\sqrt{3}}\right) + C$$

$$— = \theta - \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C = \theta - \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan\left(\frac{\theta}{2}\right)+1}{\sqrt{3}}\right) + C$$

4 Find: (b) $\int \frac{\cos\theta}{2-\cos\theta}\,d\theta = \int \frac{\cos\theta - 2 + 2}{2-\cos\theta}\,d\theta = \int \frac{\cos\theta - 2}{2-\cos\theta} + \frac{2}{2-\cos\theta}\,d\theta$

$= \int (-1) + \frac{2}{2-\cos\theta}\,d\theta \qquad \cos\theta = \frac{1-t^2}{1+t^2} \qquad d\theta = \frac{2}{1+t^2}\,dt$

$= -\theta + \int \frac{2}{2-\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{(1+t^2)}\,dt = -\theta + \int \frac{4\,dt}{2(1+t^2)-(1-t^2)}$

$= -\theta + \int \frac{4\,dt}{3t^2+1} = -\theta + \frac{4}{3}\int \frac{dt}{t^2+(1/\sqrt{3})^2} + C$

$= -\theta + \frac{4}{3}\left[\int \frac{(1/\sqrt{3})\,dt}{t^2+(1/\sqrt{3})^2}\right] \times \sqrt{3} + C$

$= -\theta + \frac{4\sqrt{3}}{3}\tan^{-1}(\sqrt{3}t) + C$

$= -\theta + \frac{4\sqrt{3}}{3}\tan^{-1}\left(\sqrt{3}\tan\left(\frac{\theta}{2}\right)\right) + C$

4 Find: **(e)** $\int \frac{\cos x}{\sin x + 1} dx$

$$\int \frac{\cos x}{\sin x + 1} dx = \int \frac{(\sin x + 1)'}{(\sin x + 1)} dx$$

$$= \ln |\sin x + 1| + C$$

NOT too difficult, this one

4 Find: (g) $\int \frac{\tan x}{1+\cos x}\,dx$

$\cos x = \frac{1-t^2}{1+t^2}$ $\quad \tan x = \frac{2t}{1-t^2}$

$dx = \frac{2}{1+t^2}\,dt$

$$\int \frac{\tan x}{1+\cos x}\,dx = \int \frac{\frac{2t}{1-t^2}}{1+\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{\frac{2t}{1-t^2}}{1+t^2+(1-t^2)} \times 2dt = \int \frac{\frac{2t}{1-t^2}}{2+t^2-t^2} \times 2\,dt$$

$$= \int \frac{\frac{2t}{1-t^2}}{2} \times 2\,dt = \int \frac{2t}{1-t^2}\,dt$$

$$= -\int \frac{(-2t)}{1-t^2}\,dt = -\int \frac{(1-t^2)'}{1-t^2}\,dt$$

$$= -\ln|1-t^2| + C$$

$$= -\ln\left|1-\tan^2\frac{x}{2}\right| + C$$

6 Use an appropriate substitution of the type $t = \tan x$ to find : **(a)** $\int \frac{dx}{1+\sin 2x}$

let $X = 2x$ so $\frac{dX}{dx} = 2$ and $dX = 2\,dx$

$$\int \frac{dx}{1+\sin 2x} = \frac{1}{2}\int \frac{1}{1+\sin X} \times dX$$

Now $\sin X = \frac{2t}{1+t^2}$ $\quad dX = \frac{2}{1+t^2}\,dt$

$$\text{—} = \frac{1}{2}\int \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$\text{—} = \frac{1}{2}\int \frac{2}{1+t^2+2t}\,dt = \int \frac{1}{(1+t)^2}\,dt$$

let $u = 1+t$ $\quad \frac{du}{dt} = 1$ so $du = dt$

$$\text{—} = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{u^{-1}}{(-1)} + C = -\frac{1}{u} + C$$

$$\text{—} = -\frac{1}{(1+t)} + C = -\frac{1}{1+\tan\left(\frac{X}{2}\right)} + C$$

$$\text{So } \int \frac{dx}{1+\sin 2x} = -\frac{1}{1+\tan\left(\frac{2x}{2}\right)} + C$$

$$\text{——} = -\frac{1}{1+\tan x} + C$$

6 Use an appropriate substitution of the type $t = \tan x$ to find (b) $\int \frac{\tan 2x}{1+\cos 2x}\,dx$

let $X = 2x \qquad dX = 2\,dx$

$$\int \frac{\tan 2x}{1+\cos 2x}\,dx = \frac{1}{2}\int \frac{\tan X}{1+\cos X}\,dX$$

Now $\cos X = \frac{1-t^2}{1+t^2}$, $\tan X = \frac{2t}{1-t^2}$ and $dX = \frac{2}{1+t^2}\,dt$

$$— = \frac{1}{2}\int \frac{\frac{2t}{1-t^2}}{1+\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{(1+t^2)}\,dt$$

$$— = \frac{1}{2}\int \frac{\frac{2t}{1-t^2}}{1+t^2+1-t^2} \times 2\,dt = \int \frac{\frac{2t}{1-t^2}}{2}\,dt$$

$$— = \frac{1}{2}\int \frac{2t}{1-t^2}\,dt = -\frac{1}{2}\int \frac{-2t}{1-t^2}\,dt$$

$$— = -\frac{1}{2}\int \frac{(1-t^2)'}{(1-t^2)}\,dt$$

$$— = -\frac{1}{2}\ln|1-t^2| + C$$

$$— = -\frac{1}{2}\ln\left|1-\tan^2\left(\frac{X}{2}\right)\right| + C$$

$$— = -\frac{1}{2}\ln\left|1-\tan^2\left(\frac{2x}{2}\right)\right| + C$$

$$— = -\frac{1}{2}\ln\left|1-\tan^2 x\right| + C$$

THE SUBSTITUTION $t = \tan(A/2)$

7 Use the substitution $t = \tan\frac{x}{2}$ to find $\int \frac{d\theta}{1+\cos\theta+\sin\theta}$.

$\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$ $\quad d\theta = \frac{2}{1+t^2}\,dt$

$$\int \frac{d\theta}{1+\cos\theta+\sin\theta} = \int \frac{1}{1+\left(\frac{1-t^2}{1+t^2}\right)+\frac{2t}{1+t^2}} \times \frac{2}{(1+t^2)}\,dt$$

$$\text{———} = \int \frac{2}{1+t^2+(1-t^2)+2t}\,dt = \int \frac{2}{2t+2}\,dt$$

$$\text{———} = \int \frac{1}{t+1}\,dt = \ln|t+1| + C$$

$$\text{———} = \ln\left|\tan\left(\frac{\theta}{2}\right)+1\right| + C$$

9 Use the substitution $t = \tan\theta$ to find the exact value of $\int_0^{\frac{\pi}{4}} \frac{d\theta}{2+\sin 2\theta}$.

First I look for $\int \frac{d\theta}{2+\sin 2\theta}$

let $X = 2\theta$ so $\frac{dX}{d\theta} = 2$ $\quad dX = 2d\theta \quad d\theta = \frac{1}{2}dX$

$$\int \frac{d\theta}{2+\sin 2\theta} = \frac{1}{2}\int \frac{dX}{2+\sin X}$$

Now $\sin X = \frac{2t}{1+t^2}$ $\quad dX = \frac{2}{1+t^2}dt$

$$\text{—} = \frac{1}{2}\int \frac{1}{2+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}dt = \int \frac{1}{2(1+t^2)+2t}dt$$

$$\text{—} = \frac{1}{2}\int \frac{dt}{t^2+t+1} = \frac{1}{2}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2 - \frac{1}{4}+1}$$

$$\text{—} = \frac{1}{2}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}\left[\int \frac{\sqrt{3}/2}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt\right] \times \frac{2}{\sqrt{3}}$$

$$\text{—} = \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right] = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C$$

$$\text{—} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2 \times \tan x + 1}{\sqrt{3}}\right) + C$$

$$\therefore \int_0^{\pi/4} \frac{d\theta}{2+\sin 2\theta} = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x+1}{\sqrt{3}}\right)\right]_0^{\pi/4} = \frac{1}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$\text{—} = \frac{1}{\sqrt{3}}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{\sqrt{3}}\left[\frac{1}{3}-\frac{1}{6}\right] = \frac{\pi}{6\sqrt{3}} = \frac{\sqrt{3}\pi}{18}$$

10 Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{12\sin x - 5\cos x + 13}$.

First I look for the primitive.

$$\int \frac{dx}{12\sin x - 5\cos x + 13} = \int \frac{1}{12\left(\frac{2t}{1+t^2}\right) - 5\left(\frac{1-t^2}{1+t^2}\right) + 13} \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{24t - 5 + 5t^2 + 13 + 13t^2}\,dt$$

$$= \int \frac{2}{18t^2 + 24t + 8}\,dt = \int \frac{dt}{9t^2 + 12t + 4}$$

$9t^2 + 12t + 4 = (3t+2)^2$ so

$$= \int \frac{dt}{(3t+2)^2}$$

let $X = 3t + 2$ $\quad \frac{dX}{dt} = 3$

so $dt = \frac{dX}{3}$

$$= \frac{1}{3}\int \frac{dX}{X^2} = \frac{1}{3}\int X^{-2}\,dX = \frac{1X^{-1}}{3(-1)} + C = -\frac{1}{3X} + C$$

$$= -\frac{1}{3(3t+2)} + C = -\frac{1}{3\left(3\tan\frac{x}{2} + 2\right)} + C$$

$$\therefore \int_{\pi/3}^{\pi/2} \frac{dx}{12\sin x - 5\cos x + 13} = \left[-\frac{1}{3\left(3\tan\frac{x}{2} + 2\right)}\right]_{\pi/3}^{\pi/2}$$

$$= -\frac{1}{3}\left[\frac{1}{3\tan\left(\frac{\pi}{4}\right) + 2} - \frac{1}{3\tan\left(\frac{\pi}{6}\right) + 2}\right]$$

$$= -\frac{1}{3}\left[\frac{1}{5} - \frac{1}{3 \times \frac{1}{\sqrt{3}} + 2}\right] = \frac{1}{3}\left[\frac{1}{\sqrt{3}+2} - \frac{1}{5}\right] = \frac{1}{3}\left[\frac{5 - \sqrt{3} - 2}{5(\sqrt{3}+2)}\right]$$

$$= \frac{3 - \sqrt{3}}{15(\sqrt{3}+2)} = \frac{(3-\sqrt{3})(\sqrt{3}-2)}{15(3-4)}$$

$$= \frac{3\sqrt{3} - 6 - 3 + 2\sqrt{3}}{-15} = \frac{9 - 5\sqrt{3}}{15}$$