OTHER REPRESENTATIONS OF COMPLEX NUMBERS - EULER'S FORMULA

Euler's formula

So far, we learnt that a complex number of modulus 1 can be represented as $\cos x + i \sin x$
In fact, this formula can be substituted with the term e^{ix}; this is known as Euler's formula:

$$\boldsymbol{e^{ix} = \cos x + i \sin x}$$

Demonstration of Euler's formula

Consider the function $f(x) = cos\, x + i\, sin\, x$ which we can differentiate as if these were real numbers:

$$\frac{df(x)}{dx} = -\,sin\, x + i\, cos\, x$$

$$\frac{df(x)}{dx} = i^2\, sin\, x + i\, cos\, x$$

$$\frac{df(x)}{dx} = i\,(cos\, x + i\, sin\, x)$$

$$\frac{df(x)}{dx} = i\, f(x)$$

We know from the study of differential equations that the D.E. $\frac{df}{dx} = k\, f$ has for general solution:

$$f(x) = A\, e^{kx}$$

where A is a constant, that can be found using initial conditions.

Therefore the D.E. $\frac{df}{dx} = i\, f$ has for general solution: $f(x) = A\, e^{ix}$

Further, for $x = 0$, we have $\cos 0 + i \sin 0 = 1$, i.e. $f(0) = 1$
Therefore $A\, e^{i0} = 1$ so $A = 1$ i.e. $f(x) = e^{ix}$

$$e^{ix} = \cos x + i \sin x \quad \text{(Euler's formula)}$$

Particularly, for $x = \pi$, we obtain: $e^{i\pi} = \cos \pi + i \sin \pi = -1 + i \times 0 = -1$

So $e^{i\pi} + 1 = 0$ which is known as "*Euler's identity*"

Euler's identity is considered to be one the most beautiful and famous equations as it links five fundamental mathematical constants e, π, i, 1 (multiplicative identity) and 0 (additive identity), and also as it includes the operations of addition, multiplication and exponentiation.

Euler's formula $e^{ix} = \cos x + i \sin x$ applies to complex numbers of modulo 1 (i.e. $|z| = 1$)
We have shown that complex numbers with modulo different of 1 (i.e. $|z| \neq 1$) can be written as:

$z = r(\cos\theta + i \sin\theta)$ or using Euler's formula $$z = re^{i\theta}$$

from there we can redemonstrate de Moivre's theorem [i.e., if $z = r(\cos\theta + i \sin\theta)$ then $z^n = r^n(\cos(n\theta) + i \sin(n\theta))$] which was otherwise demonstrated by induction in the previous lesson.

Demonstration of de Moivre's theorem using Euler's formula

Let $z = r(\cos\theta + i \sin\theta) = re^{i\theta}$

$z^n = \left(re^{i\theta}\right)^n = r^n\left(e^{i\theta}\right)^n = r^n e^{in\theta} = r^n[\cos(n\theta) + i \sin(n\theta)]$

So indeed: $[r(\cos\theta + i \sin\theta)]^n = r^n[\cos(n\theta) + i \sin(n\theta)]$

Example 13

Write each complex number in both polar and Cartesian form.

(a) $e^{\frac{i\pi}{6}}$ **(b)** $e^{\frac{-i\pi}{3}}$ **(c)** $e^{\frac{3\pi i}{4}}$ **(d)** $e^{\frac{-5\pi i}{6}}$ **(e)** $e^{-i\pi}$ **(f)** $e^{1+\frac{i\pi}{6}}$

Solution

(a) $e^{\frac{i\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

(b) $e^{\frac{-i\pi}{3}} = \cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right) = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

(c) $e^{\frac{3\pi i}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = \frac{1}{\sqrt{2}}(-1+i)$

(d) $e^{\frac{-5\pi i}{6}} = \cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right) = -\cos\frac{\pi}{6} - i\sin\frac{\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

(e) $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -\cos 0 + i\sin 0 = -1$

(f) $e^{1+\frac{i\pi}{6}} = e \times e^{\frac{i\pi}{6}} = e\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = e\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{e\sqrt{3}}{2} + \frac{e}{2}i$

Example 14

Write each complex number in the form $re^{i\theta}$, giving any decimal answers correct to two decimal places.

(a) $3(\cos 2 + i\sin 2)$ **(b)** $-1 + i\sqrt{3}$ **(c)** $2 + 3i$ **(d)** $2(\cos 1.5 - i\sin 1.5)$ **(e)** $-3 - 3i$

Solution

(a) $3(\cos 2 + i\sin 2) = 3e^{2i}$

(b) $-1 + i\sqrt{3} = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2e^{\frac{2\pi i}{3}}$

(c) $2 + 3i = \sqrt{13}\left(\frac{2}{\sqrt{13}} + \frac{3}{\sqrt{13}}i\right) = \sqrt{13}(\cos\theta + i\sin\theta)$ where $\theta = \tan^{-1}\left(\frac{3}{2}\right) \approx 0.98$

$= \sqrt{13}e^{0.98i}$

(d) $2(\cos 1.5 - i\sin 1.5) = 2(\cos(-1.5) + i\sin(-1.5)) = 2e^{-1.5i}$

(e) $-3 - 3i = 3(-1-i) = 3\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 3\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right) = 3\sqrt{2}e^{\frac{-3\pi i}{4}}$

Example 15

(a) Write $z = 1 + i$ in the form $re^{i\theta}$.

(b) Hence find the following in both polar form and Cartesian form.

(i) z^2 **(ii)** z^3 **(iii)** z^4 **(iv)** $\sqrt{z}$ **(v)** z^{-1}

Solution

(a) $z = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{\frac{i\pi}{4}}$

(b) **(i)** $z^2 = \left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^2 = 2e^{\frac{i\pi}{2}} = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 2i$

(ii) $z^3 = \left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^3 = 2\sqrt{2}e^{\frac{3\pi i}{4}} = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -2 + 2i$

This answer could also have been obtained using $z^3 = z^2 \times z = 2i(1 + i) = -2 + 2i$.

(iii) $z^4 = \left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^4 = 4e^{i\pi} = 4(\cos\pi + i\sin\pi) = -4$

(iv) $\sqrt{z} = \left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{\frac{1}{2}} = \sqrt[4]{2}e^{\frac{i\pi}{8}} = \sqrt[4]{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right) = \sqrt[4]{2}(0.9239 + 0.3827i) = 1.099 + 0.4204i$

(v) $z^{-1} = \left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{-1} = \frac{1}{\sqrt{2}}e^{\frac{-i\pi}{4}} = \frac{1}{\sqrt{2}}\left(\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \frac{1}{2} - \frac{1}{2}i$

Geometrical representation of products of complex numbers – Consolidation and Summary

Multiplication of a complex number z by a real number k

- $arg(kz) = arg(k) + arg(z)$
  - if $k > 0$ then $arg(kz) = arg(z)$
  - if $k < 0$ then $arg(k) = \pi$ and so $arg(kz) = \pi + arg(z)$ however we need to subtract 2π to find the principal argument [which is between $(-\pi)$ (not inclusive) and π (inclusive) as noted before]
- $|kz| = |k| \times |z|$ there is a scaling factor of $|k|$. If $k < 0$ then the direction from the origin O to the point representing kz is opposite to the direction from O to the point representing z

Multiplication of a complex number z by i

- $arg(iz) = arg(i) + arg(z) = \frac{\pi}{2} + arg(z)$
- $|iz| = |i| \times |z| = 1 \times |z| = |z|$ as $|i| = 1$

Hence multiplication by i causes an anticlockwise rotation by $\frac{\pi}{2}$ about the origin O, with no change to modulus.

Multiplication of a complex number z by ki where k is a real number.

- This combines the two cases above
- Rotate by $\frac{\pi}{2}$ about the origin O, and then scale by a factor $|k|$ remembering also to reverse the direction if k is negative.

Multiplication of a complex number $z_1 = r_1\, e^{i\theta_1}$ by another complex number $z_2 = r_2\, e^{i\theta_2}$

- $arg(z_1\, z_2) = arg(z_1) + arg(z_2)$ (although $arg(z_1) + arg(z_2)$ is one value of $arg(z_1 z_2)$, but not necessarily the principal value; we may have to add or subtract a multiple of 2π to obtain the principal argument)
- $|z_1\, z_2| = |z_1| \times |z_2|$
- So to multiply by $r\, e^{i\theta}$, we rotate by θ anticlockwise about O, and then we scale by a factor r

Example 16

Given $z_1 = 2e^{\frac{i\pi}{6}}$, $z_2 = 3e^{\frac{-i\pi}{3}}$ and $z_3 = e^{\frac{3i\pi}{4}}$, find the polar form for each of the following.

(a) $z_1 \times z_2$ **(b)** $z_2 \times z_3$ **(c)** $z_1{}^2 \times z_2$ **(d)** $\frac{z_1}{z_2}$

(e) $\frac{z_2}{z_3}$ **(f)** $\frac{z_1{}^2 \times z_2}{z_3}$ **(g)** On the Argand diagram, plot z_1, z_2 and $z_1 \times z_2$.

(h) On the Argand diagram, plot z_2, z_3 and $\frac{z_2}{z_3}$.

Solution

(a) $z_1 \times z_2 = 2e^{\frac{i\pi}{6}} \times 3e^{\frac{-i\pi}{3}} = 6e^{\frac{i\pi}{6} + \frac{-i\pi}{3}} = 6e^{\frac{-i\pi}{6}}$

(b) $z_2 \times z_3 = 3e^{\frac{-i\pi}{3}} \times e^{\frac{3i\pi}{4}} = 3e^{\frac{-i\pi}{3} + \frac{3i\pi}{4}} = 3e^{\frac{5i\pi}{12}}$

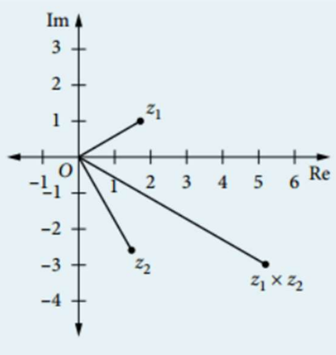
(c) $z_1{}^2 \times z_2 = 2^2 e^{\frac{2i\pi}{6}} \times 3e^{\frac{-i\pi}{3}} = 12e^{\frac{i\pi}{3} + \frac{-i\pi}{3}} = 12e^0 (= 12)$

(d) $\frac{z_1}{z_2} = \frac{2e^{\frac{i\pi}{6}}}{3e^{\frac{-i\pi}{3}}} = \frac{2}{3} e^{\frac{i\pi}{6} - \frac{-i\pi}{3}} = \frac{2}{3} e^{\frac{i\pi}{2}} \left(= \frac{2}{3} i\right)$

(e) $\frac{z_2}{z_3} = \frac{3e^{\frac{-i\pi}{3}}}{e^{\frac{3i\pi}{4}}} = 3e^{\frac{-i\pi}{3} - \frac{3i\pi}{4}} = 3e^{\frac{-13i\pi}{12}} = 3e^{\frac{11i\pi}{12}}$

(f) $\frac{z_1{}^2 \times z_2}{z_3} = \frac{2^2 e^{\frac{2i\pi}{6}} \times 3e^{\frac{-i\pi}{3}}}{e^{\frac{3i\pi}{4}}} = \frac{12e^0}{e^{\frac{3i\pi}{4}}} = 12e^{\frac{-3i\pi}{4}}$

(g)



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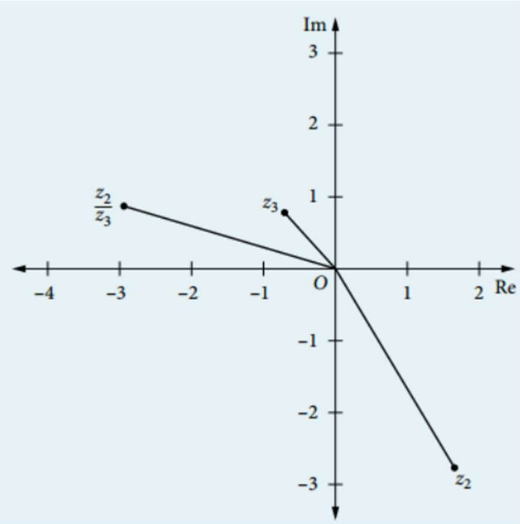
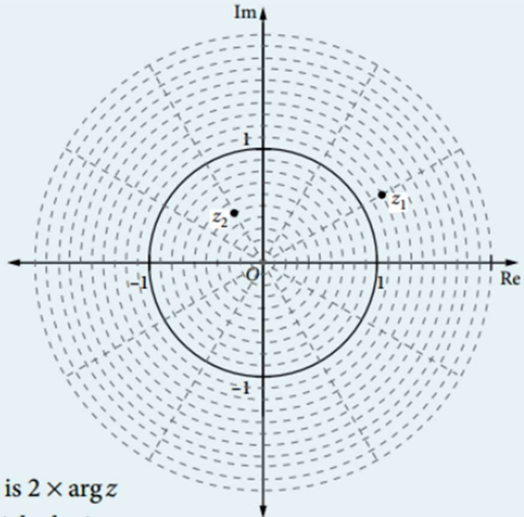


Example 17

The Argand diagram at right shows the unit circle as well as points representing the complex numbers z_1 and z_2.

For **(a)** $z = z_1$ and **(b)** $z = z_2$, mark points A, B, C, D, E, F, G to represent $\overline{z}$, $2z$, $-z$, iz, $-\frac{1}{2}iz$, z^2 and $(1+\sqrt{3}i)z$.



Solution

A: $\overline{z}$ is the reflection of z in the real axis

B: $2z$ is z scaled by a factor of 2

C: $-z$ is z scaled by a factor of -1 (i.e. reflected back through O)

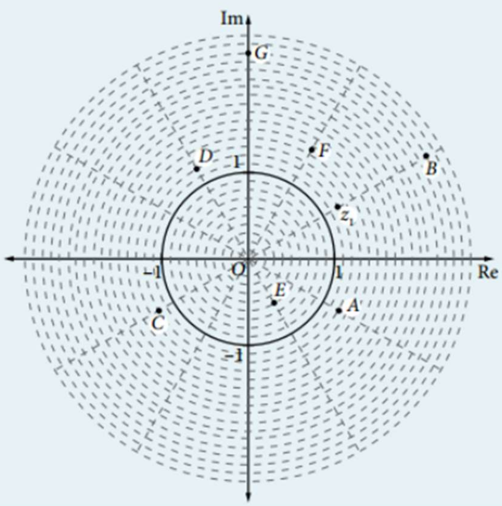
D: iz is z rotated by $\frac{\pi}{2}$ anticlockwise about O

E: $-\frac{1}{2}iz$ is iz scaled by a factor of $-\frac{1}{2}$

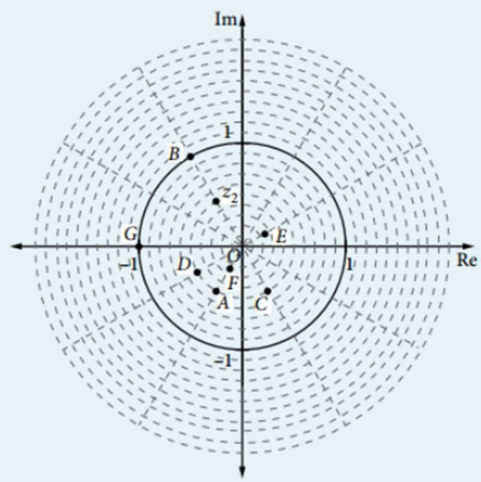
F: z^2 has a modulus that is $(\text{mod}\, z)^2$ and an argument that is $2 \times \arg z$

G: $1+\sqrt{3}i = 2\,\text{cis}\,\frac{\pi}{3}$, so $(1+\sqrt{3}i)z$ is found by rotating anticlockwise by $\frac{\pi}{3}$ and then doubling the modulus.

(a)



(b)



Example 18

Let $OABC$ be a square on an Argand diagram where O is the origin. The points A and C represent the complex numbers z and iz respectively.

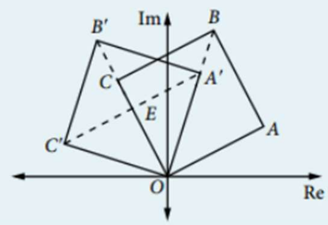
(a) Find the complex number represented by B.

(b) The square is now rotated anticlockwise 45° about O to form $OA'B'C'$. Find the complex numbers represented by A', B' and C'.

(c) E is the point of intersection of the diagonals of the square $OA'B'C'$. What complex number does E represent?

Solution

(a) B represents $z + iz$ (completion of the parallelogram represents the sum)



(b) **Method 1**

A' is formed by rotating A anticlockwise by 45° about O.

Hence A' represents $z \times 1(\cos 45° + i \sin 45°)$ $= \frac{z}{\sqrt{2}}(1 + i)$

B' represents $(z + iz) \times 1(\cos 45° + i \sin 45°)$ $= \frac{z}{\sqrt{2}}(1 + i)^2 = \frac{z}{\sqrt{2}} \times 2i = \sqrt{2}\, iz$

Method 2

In a square, the length of the diagonal is $\sqrt{2}$ times the length of a side. Also, the diagonals are inclined at 45° to the sides. Hence, when A is rotated by 45° to A', A' is the point on the diagonal OB which is $\frac{1}{\sqrt{2}}$ from O.

Thus A' represents the number $\frac{1}{\sqrt{2}} \times (z + iz)$ $= \frac{z}{\sqrt{2}}(1 + i)$

B' is the point along the extension of OC such that $OB' = \sqrt{2} \times OC$.

Hence B' represents $\sqrt{2} \times iz = \sqrt{2}\, iz$

By either method, similarly C' is:
$iz \times 1(\cos 45° + i \sin 45°) = \frac{z}{\sqrt{2}}(-1 + i)$

(c) The diagonals of a square bisect each other, so E is the midpoint of OB'.

Hence E represents: $\frac{1}{2} \times \sqrt{2}\, iz$