# MATHEMATICAL INDUCTION, HARDER QUESTIONS

Use mathematical induction to prove the following results.

**1** Prove that $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ for $n$ a positive integer.

Step 1 For $n=1$ $\sum_{r=1}^{1} \frac{1}{r(r+1)} = \frac{1}{1 \times 2}$ whereas $\frac{1}{1+1} = \frac{1}{2}$ too.

So it's true for $n=1$.

Step 2 : we assume it's true for $n=k$ $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

then : $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$\therefore$ it's true for $n=k+1$ if it's true for $n=k$.

Step 3. It's true for $n=1$

it's true for $n=k+1$ if it's true for $n=k$.

$\therefore$ it's true, by induction, for $n \geq 1$ ($n$ integer)

# MATHEMATICAL INDUCTION, HARDER QUESTIONS

**3** Prove that $n^2 + 2n$ is divisible by 8 if $n$ is an even integer.

Step 1 $\quad n = 2 \qquad n^2 + 2n = 2^2 + 2 \times 2 = 8$ divisible by 8 indeed.

Step 2 $\quad$ Assume it's true for $n = k$, i.e. $\exists\, q \in \mathbb{N}$ such that $k^2 + 2k = 8q$

In that case: $(k+2)^2 + 2(k+2) = k^2 + 4k + 4 + 2k + 4$

$$\_\_\_\_ = 8q + 4k + 8$$

But $k$ is an even integer, by assumption, so $\exists\, m \in \mathbb{N}$ such that $k = 2m$, i.e.

$$LHS = 8q + 8m + 8 = 8[q + m + 1]$$

$\therefore$ it's true for $(k+2)$ if it's true for $k$.

Step 3. $\quad$ it's true for $n = 2$ (actually also true for $n = 0$)

it's true for $(k+2)$ if it's true for $k$.

$\therefore$ it's true for any even integer, by induction.

**4** Prove that $3^{4n} - 1$ is divisible by 80 for $n$ a positive integer.

Step 1 $\quad n=1 \quad 3^{4\times 1} - 1 = 81 - 1 = 80$ divisible by 80 indeed.

Step 2 $\quad$ Assume it's true for $n = k$. $\exists q \in \mathbb{N} / 3^{4k} - 1 = 80q$.

In that case $\quad 3^{4(k+1)} - 1 = 3^{4k} \times 3^4 - 1$

$$= 81 \times 3^{4k} - 1$$

$$= 81 \times [80q + 1] - 1$$

$$= 80 \times 81q + 81 - 1$$

$$= 80 \times 81q + 80 = 80[81q + 1]$$

$\therefore$ it's true for $k+1$

Step 3 $\quad$ it's true for $n = 1$ (actually also true for $n=0$)

it's true for $(k+1)$ if it's true for $k$.

$\therefore$ By induction, it's true for any $n$ positive integer.

**9** Prove that $\sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right) = \log(n+1)$.

Step 1 $n = 1$ $\sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right) = \log\left(\frac{2}{1}\right) = \log 2$

whereas $\log(1+1) = \log 2$ too.

Step 2 Assume it's true for $n = r$, i.e. $\sum_{k=1}^{r} \log\left(\frac{k+1}{k}\right) = \log(r+1)$

In that case:

$$\sum_{k=1}^{r+1} \log\left(\frac{k+1}{k}\right) = \left[\sum_{k=1}^{r} \log\left(\frac{k+1}{k}\right)\right] + \log\left(\frac{r+2}{r+1}\right)$$

$$= \log(r+1) + \log\left(\frac{r+2}{r+1}\right)$$

$$= \log\left[\frac{(r+1)(r+2)}{(r+1)}\right] = \log(r+2).$$

$\therefore$ it's also true for $(r+1)$

Step 3. it's true for $n = 1$

it's true for $(r+1)$ if it's true for $r$

$\therefore$ by induction, it's true for any $n$ integer strictly positive.

## MATHEMATICAL INDUCTION, HARDER QUESTIONS

**11** Prove that $x^n - 1$ is divisible by $(x-1)$ for $n$ a positive integer. Use the result that $\frac{x^n - 1}{x-1} = x^{n-1} + \frac{x^{n-1}-1}{x-1}$

Step 1 $\quad n = 1 \quad x^n - 1 = x - 1$ divisible by $(x-1)$ indeed.

Step 2 Assume it's true for $n = k$

In that case $\frac{x^{k+1}-1}{x-1} = x^k + \frac{x^k - 1}{x-1}$

$$\text{———} = \underbrace{x^k}_{\text{integer.}} + \underbrace{\frac{x^k-1}{x-1}}_{\text{integer, by assumption.}}$$

$\therefore \frac{x^{k+1}-1}{x-1}$ is equal to an integer, i.e.

$(x^{k+1} - 1)$ is divisible by $(x-1)$.

Step 3 it's true for $n = 1$

it's true for $(k+1)$ if it's true for $k$.

$\therefore$ by induction, it's true for any positive integer.

**12** Prove that $\sum_{r=1}^{n} r\log\left(\frac{r+1}{r}\right) = \log\frac{(n+1)^n}{n!}$.

<u>Step 1</u> $n=1$ $\quad \sum_{r=1}^{1} r\log\left(\frac{r+1}{r}\right) = 1 \times \log\left(\frac{1+1}{1}\right) = \log 2$

whereas $\log\left[\frac{(1+1)^1}{1!}\right] = \log 2$ too.

<u>Step 2</u> Assume it's true for $n=k$.

i.e. $\sum_{r=1}^{k} r\log\left(\frac{r+1}{r}\right) = \log\left[\frac{(k+1)^k}{k!}\right]$

Then $\sum_{r=1}^{k+1} r\log\left(\frac{r+1}{r}\right) = \sum_{r=1}^{k} r\log\left(\frac{r+1}{r}\right) + (k+1)\log\left(\frac{k+2}{k+1}\right)$

$$= \log\left[\frac{(k+1)^k}{k!}\right] + (k+1)\log\left(\frac{k+2}{k+1}\right)$$

$$= \log\left[\frac{(k+1)^k}{k!}\right] + k\log\left(\frac{k+2}{k+1}\right) + \log\left(\frac{k+2}{k+1}\right)$$

$$= \log\left[\frac{(k+1)^k}{k!}\right] + \log\left[\frac{(k+2)^k}{(k+1)^k}\right] + \log\left(\frac{k+2}{k+1}\right)$$

$$= \log\left[\frac{\cancel{(k+1)^k}\,(k+2)^k}{k!\,\cancel{(k+1)^k}}\right] + \log\left(\frac{k+2}{k+1}\right)$$

$$= \log\left[\frac{(k+2)^k}{k!}\right] + \log\left(\frac{k+2}{k+1}\right)$$

$$= \log\left[\frac{(k+2)^{k+1}}{(k+1)!}\right]$$ So true for $(k+1)$

<u>Step 3</u> : true for $n=1$

True for $k+1$ if true for $k$

$\therefore$ by induction, true for any positive integer $n$