PROOF BY MATHEMATICAL INDUCTION - CHAPTER REVIEW

3 Prove by induction that $7^n - 1$ is divisible by 3 for all positive integers n.

① For $n = 1$ $\quad 7^1 - 1 = 6$ which is indeed divisible by 3.

② Assume this is true for $n = k$, i.e, $\exists q \in \mathbb{Z}$ such that

$7^k - 1 = 3q$.

In that case $7^{k+1} - 1 = 7^k \times 7 - 1$

$= [3q + 1] \times 7 - 1$

$= 3 \times 7q + 7 - 1$

$= 3 \times 7q + 6$

$= 3[7q + 2]$

So $7^{k+1} - 1$ is also divisible by 3 as 3 is a factor.

Step 3: it is true for $n = 1$

it's true for $n = k+1$ if it's true for $n = k$

$\therefore$ it's true for any positive integer.

4 Prove by induction that $6 + 24 + 60 + \ldots + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$ for all positive integers n.

① For $n = 1$ $\quad 6 = \dfrac{1(1+1)(1+2)(1+3)}{4} = 6$ indeed

So it's true for $n = 1$

Step 2 We assume it's true for $n = k$.

In that case:

$$6 + 24 + 60 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

$$\underline{\qquad\qquad} = \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4}$$

$$\underline{\qquad\qquad} = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

So it's also true for $(k+1)$ then.

Step 3 the statement is true for $n = 1$

the statement is true for $k+1$ if it's true for k.

$\therefore$ it's true for all positive integers.

7 Prove by induction that $7^n + 6^n$ is divisible by 13 for all odd positive integers n.

Step 1 $\quad n=1 \quad 7^1+6^1=13$ which is divisible by 13

Step 2 Assume the statement is true for $n=k$ (n odd), i.e. that

$\exists\, q \in \mathbb{Z}$ such that $7^k+6^k=13q$

In that case, the next odd integer would be $(k+2)$.

$$7^{k+2}+6^{k+2} = 7^k \times 49 + 6^{k+2}$$
$$= [13q - 6^k] \times 49 + 6^{k+2}$$
$$= 13q \times 49 + 6^k[-49+36]$$
$$= 13q \times 49 - 13 \times 6^k$$
$$= 13[49q - 6^k]$$

So the statement is also true for $k+2$.

Step 3 : (conclusion). The statement is true for 1

The statement is true for $(k+2)$ if it's true for k.

$\therefore (7^n+6^n)$ is divisible by 13 for all odd integers.

8 Prove by induction that $2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (n^2 + 1)n! = n(n+1)!$ for all positive integers n.

Step 1 $n=1$ $2 \times 1! = 2$ whereas $1 \times (1+1)! = 2$

So it's true for $n=1$

Step 2 Assume it's true for $n=k$.

In that case:

$2 \times 1! + 5 \times 2! + \ldots + (k^2+1)k! + [(k+1)^2+1](k+1)! = k(k+1)! + [(k+1)^2+1](k+1)!$

$= (k+1)!\,[k^2+3k+2]$

$= (k+1)!\,[(k+1)(k+2)]$

$= (k+2)!\,(k+1)$

$= (k+1)(k+2)!$

So the statement is then true for $(k+1)$.

Step 3 (conclusion): the statement is true for $n=1$

The statement is true for $(k+1)$ if it's true for k.

$\therefore$ the statement is true for all positive integers, by induction.