Example 15

For the function $f(t) = 2te^{-0.5t}$, find the value of t for which $f(t)$ has a maximum and hence calculate the maximum value. Sketch the graph of $f(t)$.

Solution

$f(t) = 2te^{-0.5t}$ Let $u = t$, $v = e^{-0.5t}$

$$f'(t) = 2\left(e^{-0.5t} + t \times \left(-\frac{1}{2}\right)e^{-0.5t}\right)$$

$$= e^{-0.5t}(2-t)$$

For stationary points, $f'(t) = 0$: $e^{-0.5t}(2-t) = 0$

But $e^{-0.5t} > 0$ for all t, so $t = 2$ is the only solution and $f(2) = \frac{4}{e}$

For $t < 2$: $f'(t) > 0$

For $t > 2$: $f'(t) < 0$

Gradient changes from positive to negative as x increases, so $\left(2, \frac{4}{e}\right)$ is a maximum turning point.

The maximum value of the function is $\frac{4}{e} \approx 1.472$

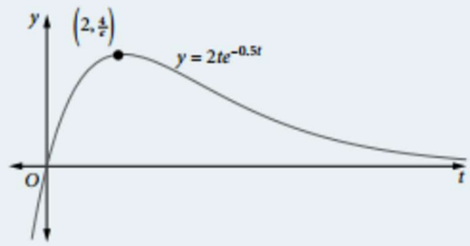
$f(t) = 0$ at $t = 0$ as $e^{-0.5t} > 0$ for all t.

$t < 0$, $f(t) < 0$ $t < 2$, $f'(t) > 0$

$t > 0$, $f(t) > 0$ $t > 2$, $f'(t) < 0$

$t \to \infty$, $f(t) \to 0$ from above

$t = 0$ is a horizontal asymptote



Example 16

Find the coordinates of any maximum or minimum turning points of the curve $y = \frac{\ln x}{e^x}$ given that $x \ln x = 1$ when $x = 1.76$.

Solution

$$y = \frac{\ln x}{e^x} = \ln x \times e^{-x}$$

$$\frac{dy}{dx} = \ln x \times -e^{-x} + \frac{1}{x} \times e^{-x}$$

$$= \frac{1 - x\ln x}{xe^x}$$

For stationary points, $\frac{dy}{dx} = 0$:

$$\frac{1 - x\ln x}{xe^x} = 0$$

$$1 - x\ln x = 0$$

$$x \ln x = 1$$

$$\therefore x = 1.76$$

$x = 1.7$: $\quad \frac{dy}{dx} = \frac{1 - 1.7\ln 1.7}{1.7e^{1.7}} \approx 0.01 > 0$

$x = 1.8$: $\quad \frac{dy}{dx} = \frac{1 - 1.8\ln 1.8}{1.8e^{1.8}} = -0.005 < 0$

The gradient changes from positive to negative on passing through the stationary point so the function has a maximum value when $x = 1.76$.

$x = 1.76$, $y = 0.097$

The coordinates of the maximum turning point are (1.76, 0.097).