VELOCITY AND ACCELERATION AS RATES OF CHANGE

1 A particle is moving in a straight line, so that its displacement x metres from a fixed point O on the line at time t seconds ($t \geq 0$) is given by $x = 2t^3 - 5t^2 - 4t$.

(a) Find the velocity and acceleration of the particle at any time t.
(b) Find the initial velocity and acceleration.
(c) When is the particle at rest?
(d) When is the acceleration zero? What is the velocity and displacement at this time?

a) $\frac{dx}{dt} = 6t^2 - 10t - 4$ $\quad\quad$ $\frac{d^2x}{dt^2} = 12t - 10$

b) At $t = 0$ $\quad$ $\frac{dx}{dt} = v(0) = 6\times0^2 - 10\times0 - 4 = -4 \text{ ms}^{-1}$

$\underbrace{\frac{d^2x}{dt^2} = a(0) = \ddot{x}(0)}_{\text{just to show different possible notations.} ☺} = 12\times0 - 10 = -10 \text{ ms}^{-2}$

c) The particle is at rest when $v = 0$, i.e. when $6t^2 - 10t - 4 = 0$

$\Delta = 10^2 - 4\times(-4)\times6 = 196 = 14^2$

So at $t_1 = \frac{10-14}{2\times6} < 0$ so impossible and $t_2 = \frac{10+14}{2\times6} = \frac{24}{12} = 2$ s.

d) $\ddot{x}$ is zero when $12t - 10 = 0$

i.e.. when $t = \frac{10}{12} = \frac{5}{6}$ s.

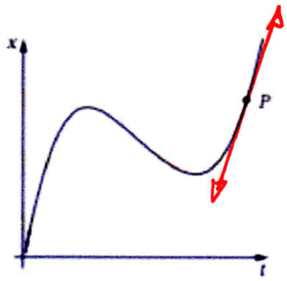
At $t = 5/6$ $\quad$ $v(5/6) = 6\times\left(\frac{5}{6}\right)^2 - 10\times\frac{5}{6} - 4 = -\frac{49}{6} \text{ ms}^{-1}$

At $t = 5/6$ $\quad$ $x(5/6) = 2\times\left(\frac{5}{6}\right)^3 - 5\times\left(\frac{5}{6}\right)^2 - 4\times\left(\frac{5}{6}\right) = -\frac{305}{54}$ m

4 The graph shows the displacement x of a particle moving along a straight line as a function of time t.

Which statement best describes the motion of the particle at the point P?

A The velocity is negative and the acceleration is positive.
B The velocity is negative and the acceleration is negative.
C The velocity is positive and the acceleration is positive.
D The velocity is positive and the acceleration is negative.



$\frac{dx}{dt} > 0$ and $\frac{d^2x}{dt^2} > 0$ (as the curve is concave up) so $\boxed{C}$

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3 A particle moves in a straight line. Its velocity v ms^{-1} at time t is given by $v = 5 - \dfrac{10}{t+1}$.

(a) Find the initial velocity.

(b) Find the acceleration of the particle when the particle is at rest.

(c) Sketch the graph of v for $t \geq 0$, showing any intercepts and asymptotes.

a) at $t = 0$ $\quad \underbrace{v(0) = \dot{x}(0)}_{\text{just to show different possible notations}} = 5 - \dfrac{10}{0+1} = 5 - 10 = -5 \text{ ms}^{-1}$ ☺

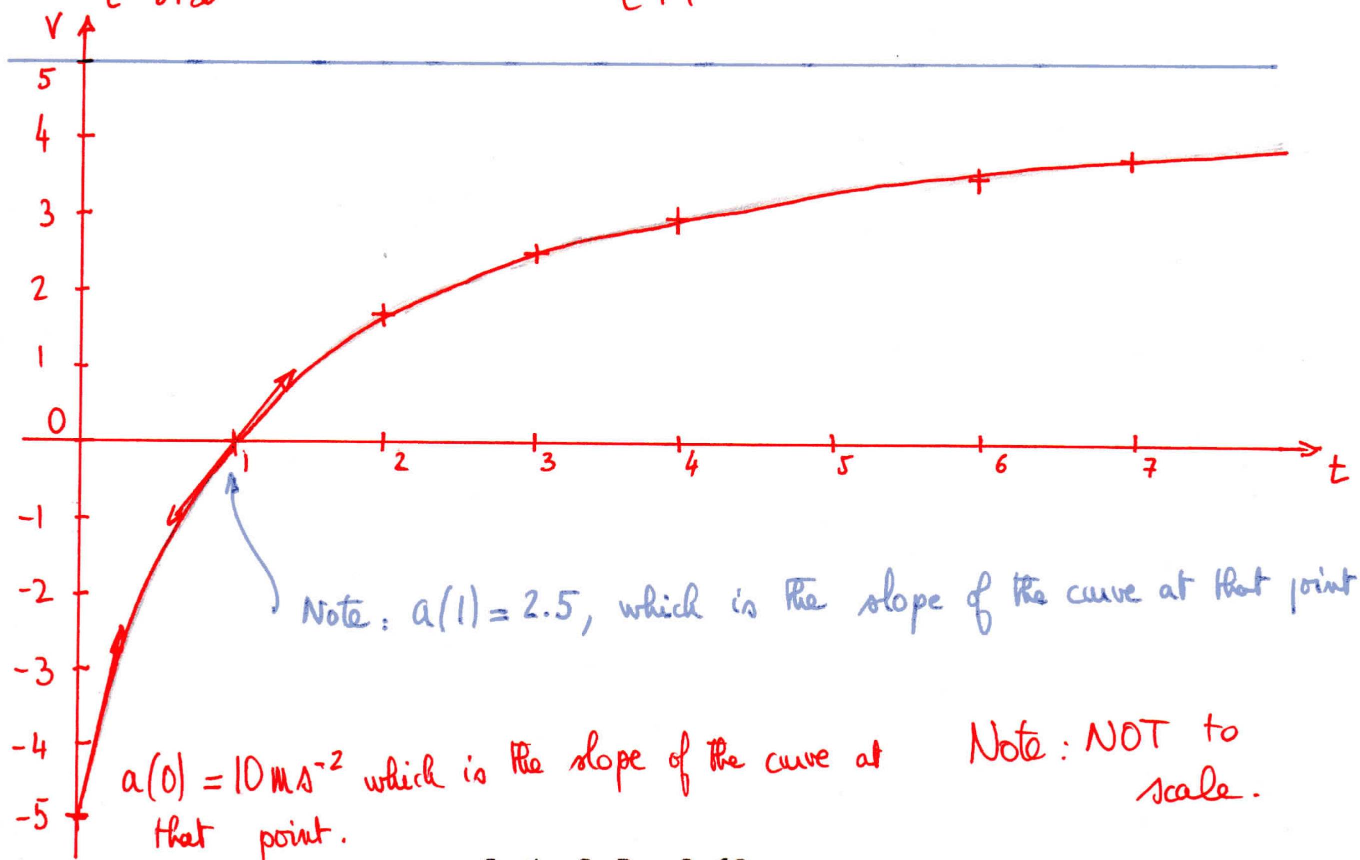
b) The particle is at rest when $v = 0$, i.e. when $5 - \dfrac{10}{t+1} = 0$

$\Leftrightarrow 5 = \dfrac{10}{t+1} \Leftrightarrow \dfrac{t+1}{10} = \dfrac{1}{5} \Leftrightarrow t+1 = \dfrac{10}{5} = 2$ so $t = 1$ s

$\underbrace{a(t) = \ddot{x}(t) = \dfrac{dv}{dt}}_{\text{different possible notations for acceleration.}} = \dfrac{d}{dt}\left(5 - \dfrac{10}{t+1}\right) = -10\dfrac{d}{dt}\left(\dfrac{1}{t+1}\right) = -10\dfrac{d}{dt}\left[(t+1)^{-1}\right]$

$a(t) = -10 \times (-1)(1+t)^{-2} = \dfrac{10}{(1+t)^2}$ $\quad$ so at $t = 1$ $\quad a(1) = \dfrac{10}{(1+1)^2} = \dfrac{10}{4} = \dfrac{5}{2}$ ms^{-2}

c) $\lim\limits_{t \to +\infty} v(t) = \lim\limits_{t \to +\infty} 5 - \dfrac{10}{t+1} = 5$ ms^{-1} (asymptote)



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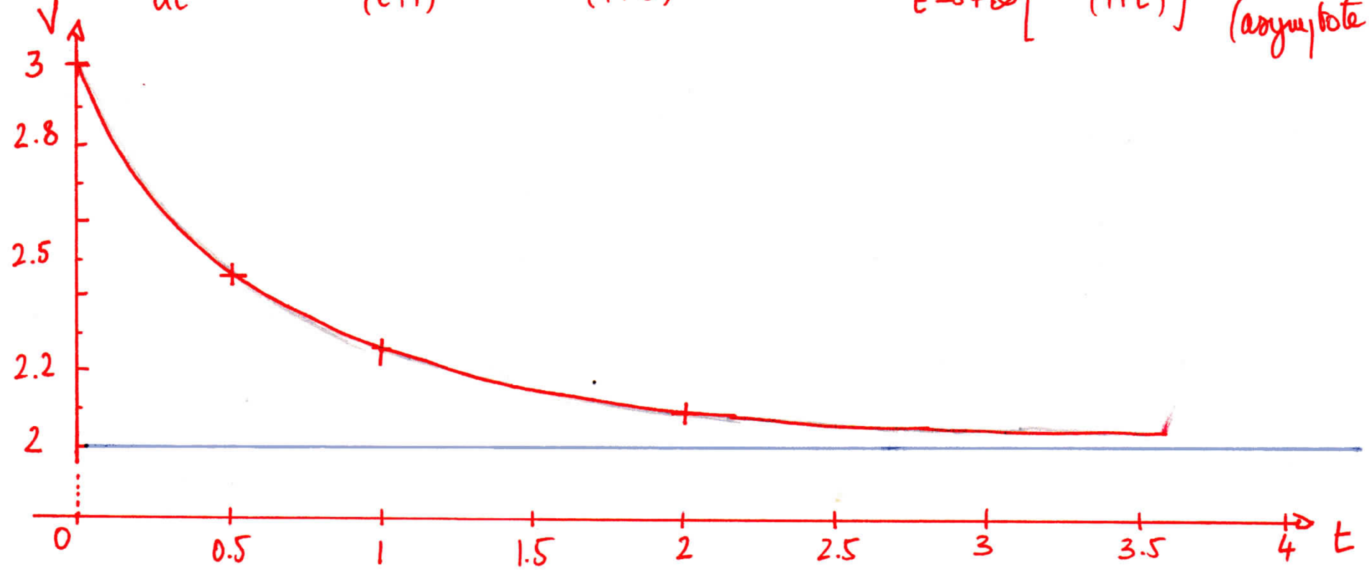
5 The displacement of a particle moving along the x-axis is given by $x = 2t - \frac{1}{t+1}$, where x is the displacement from the origin in metres, t is in seconds and $t \geq 0$.

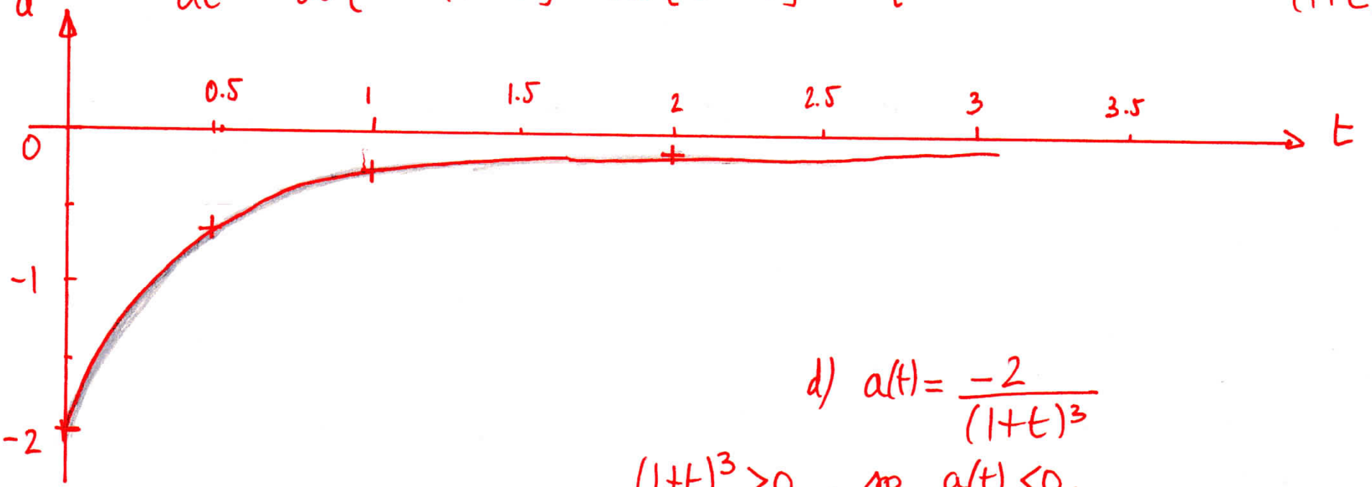
(a) Find the expression for the velocity v and draw the graph of v against t.
(b) What value does the velocity approach as t increases indefinitely?
(c) Find the expression for the acceleration a and draw the graph of a against t.
(d) Show that the acceleration of the particle is always negative.

a) $v = \frac{dx}{dt} = 2 - (-1)\frac{1}{(t+1)^2} = 2 + \frac{1}{(1+t)^2}$

b) $\lim\limits_{t \to +\infty} \left[2 + \frac{1}{(1+t)^2}\right] = 2 \text{ m s}^{-1}$ (asymptote)



c) $a = \frac{dv}{dt} = \frac{d}{dt}\left[2 + \frac{1}{(1+t)^2}\right] = \frac{d}{dt}\left[\frac{1}{(1+t)^2}\right] = \frac{d}{dt}\left[(1+t)^{-2}\right] = -2(1+t)^{-3} = \frac{-2}{(1+t)^3}$



d) $a(t) = \frac{-2}{(1+t)^3}$

$(1+t)^3 > 0$, so $a(t) < 0$.

6 A particle is moving along the x-axis. The displacement of the particle at time t is x metres. At a certain time, $v = -4 \text{ m s}^{-1}$ and $a = 3 \text{ m s}^{-2}$.

Which statement describes the motion of the particle at that time?

A The particle is moving to the left with decreasing speed.
B The particle is moving to the left with increasing speed.
C The particle is moving to the right with decreasing speed.
D The particle is moving to the right with increasing speed.

$v < 0$ so towards the left
$a > 0$ so v is increasing, i.e. getting less and less negative.
so speed decreasing.

7 A driver takes 3 hours to travel the distance between two points A and B on a country road. At time t hours after passing A, the driver's speed $v\,\mathrm{km\,h^{-1}}$ is given by $v = 60 + 40e^{-t}$.

(a) Calculate the speeds when the driver passes points A and B.

(b) Write the acceleration in terms of: **(i)** t **(ii)** v

(c) Sketch the velocity–time curve and comment on the motion for large values of t.

a) At A, $t=0$ so $v(0) = 60 + 40e^{-0} = 100$ km hr^{-1}

At B, $t=3$ so $v(3) = 60 + 40e^{-3} \simeq 62$ km hr^{-1}

b) i) $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}\left[60 + 40e^{-t}\right] = -40e^{-t}$

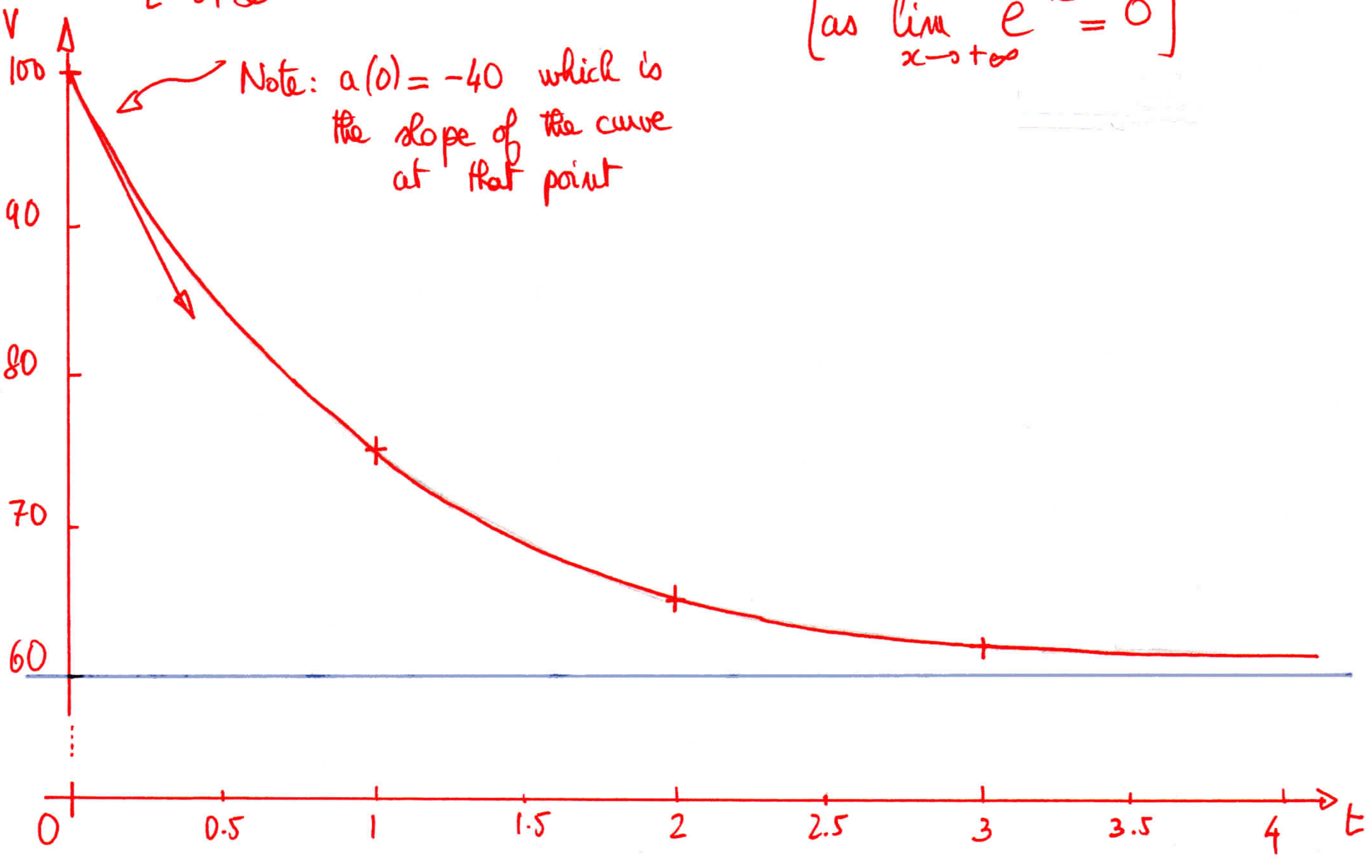
ii) $a(t) = -40e^{-t}$ but $v(t) = 60 + 40e^{-t}$

so $-40e^{-t} = 60 - v(t)$

$\therefore\ a(t) = 60 - v(t)$.

c) $\lim\limits_{t \to +\infty} v(t) = \lim\limits_{t \to +\infty} \left[60 + 40e^{-t}\right] = 60$ km hr^{-1} (asymptote)

$\left[\text{as } \lim\limits_{x \to +\infty} e^{-x} = 0\right]$



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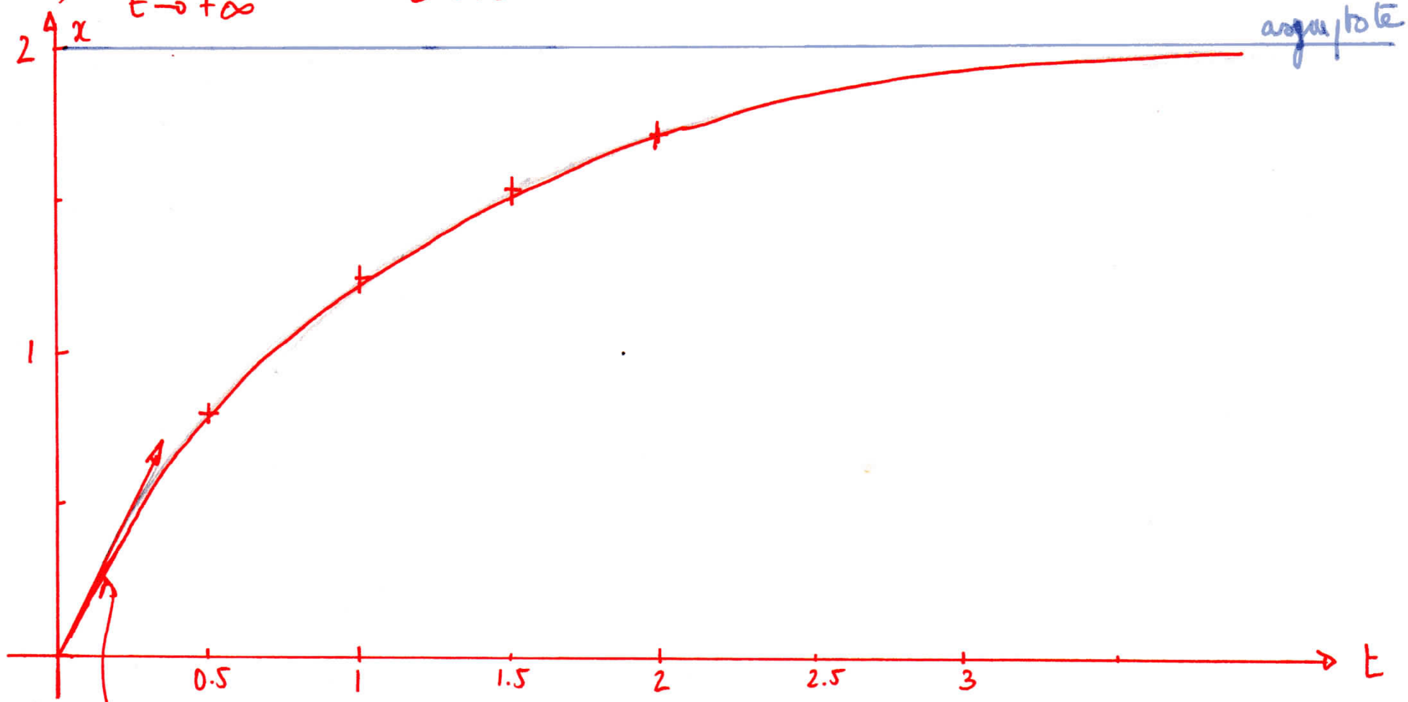
8 A particle moves in a straight line so that its displacement x from a fixed origin at any time t is given by $x(t) = 2(1 - e^{-t})$.

(a) Find $x(0)$, $\dot{x}(0)$ and $\ddot{x}(0)$. **(b)** Sketch the graph of $x(t)$. **(c)** Find t when $x(t) = 1$.

a) $x(0) = 2(1 - e^{-0}) = 0$ m $\quad \dot{x}(t) = 2e^{-t}$ $\quad$ so $\dot{x}(0) = 2e^{-0} = 2$ ms^{-1}

$\ddot{x}(t) = -2e^{-t}$ $\quad$ so $\ddot{x}(0) = -2e^{-0} = -2$ ms^{-2}

b) $\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} [2(1 - e^{-t})] = 2$ ms^{-1} (asymptote)



$\dot{x}(0) = 2$ which is the slope at that point.

c) For $x(t) = 1$, we must have $2(1 - e^{-t}) = 1$

$\Leftrightarrow 1 - e^{-t} = \frac{1}{2} \quad \Leftrightarrow e^{-t} = \frac{1}{2} \quad \Leftrightarrow -t = \ln\left(\frac{1}{2}\right) = -\ln 2$

so $t = \ln 2$

9 A body starts from O and moves in a straight line. At any time t its velocity is given by $\dot{x} = 6t - 4$. Indicate whether each statement below is correct or incorrect.

(a) $x = 3t^2 - 4t + C$ **(b)** $x = 3t^2 - 4t$ **(c)** $\ddot{x} = 3t^2 - 4t$ **(d)** $\ddot{x} = 6$

(a) $\dot{x} = 6t - 4$ So TRUE

(b) possible, but misses some solutions

(c) NO

(d) TRUE

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10 A body starts from O and moves in a straight line. At any time t, its velocity is $t^2 - 4t^3$. Find, in terms of t:

(a) the displacement x (b) the acceleration.

a) To get $x(t)$ from $v(t) = t^2 - 4t^3$, we need to find the function such that $\dfrac{dx}{dt} = t^2 - 4t^3$ (i.e. go the opposite way to differentiation)

this is called "integrating" $x(t) = \dfrac{t^3}{3} - 4\dfrac{t^4}{4} + C$

so $x(t) = \dfrac{t^3}{3} - t^4 + C$

Further, we know that at $t = 0$, $x(0) = 0$ so $C = 0$.

i.e $x(t) = \dfrac{t^3}{3} - t^4$

b) $a(t) = \dfrac{d\,v(t)}{dt} = \dfrac{d}{dt}\left[t^2 - 4t^3\right] = 2t - 12t^2 = 2t\left[1 - 6t\right]$

11 The velocity v m s^{-1} at time t seconds ($t \geq 0$) of a body moving in a straight line is given by $v = 6t^2 + 6t - 12$. Find the acceleration at any time t.

$v(t) = 6t^2 + 6t - 12$

$\underbrace{a(t) = \ddot{x}(t) = \dfrac{dv}{dt}}_{\text{different possible notations for } a(t)} = 12t + 6$

different possible notations for a(t) ☺

12 A particle is projected vertically upwards from a point O with an velocity of $25\,\mathrm{m\,s^{-1}}$ and a downward acceleration of $10\,\mathrm{m\,s^{-2}}$.

(a) Find its velocity and height above O at any time t.

(b) What maximum height does the particle reach?

(c) At what time has its velocity been reduced to half the velocity of projection?

a) $a(t) = -10\ ms^{-2} = \frac{dv(t)}{dt}$

we "integrate" (opposite of "differentiate")

so $v(t) = -10t + C$

At $t=0$, $v(0) = 25$, so $25 = -10 \times 0 + C$

$\therefore C = 25\ ms^{-1}$

And so $v(t) = -10t + 25$

b) The particle reaches its maximum height when $v(t) = 0$,

i.e. when $-10t + 25 = 0 \iff 10t = 25$ $\boxed{t = 2.5}$

c) $v(t) = \frac{25}{2}$ when $\frac{25}{2} = -10t + 25$

$\iff 10t = 25 - 12.5 = 12.5$

so at $t = 1.25\ s$.

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13 A body is projected vertically upwards with an initial velocity of $30\,\text{m s}^{-1}$. It rises with a deceleration of $10\,\text{m s}^{-2}$.

(a) Find its velocity at any time t.
(b) Find its height h m above the point of projection at any time t.
(c) Find the greatest height reached.
(d) Find the time taken to return to the point of projection.

a) $a(t) = -10 \text{ ms}^{-2} = \dfrac{dv}{dt}$

we "integrate" (opposite of "differentiate")

so $v(t) = -10t + C$

At $t=0$, $v(0) = 30$ so $30 = -10 \times 0 + C$

$\therefore C = 30$

$\therefore v(t) = -10t + 30$

b) $v(t) = \dfrac{dx}{dt} = -10t + 30$

we "integrate"

so $x(t) = -10\dfrac{t^2}{2} + 30t + C$

At $t=0$ $x(0) = 0$, so $0 = -\dfrac{10 \times 0^2}{2} + 30 \times 0 + C$

$\therefore C = 0$

$x(t) = -5t^2 + 30t$

c) The greatest height is reached when $v(t) = 0$, i.e.

$-10t + 30 = 0 \iff t = 3\text{ s}$.

At $t = 3$, $x(3) = -5 \times 3^2 + 30 \times 3 = 45\text{ m}$

which is the greatest height reached

d) $x(t) = -5t^2 + 30t = 5t\,[6 - t]$

so $x(t) = 0$ at $t = 0$ (when it's lauched) and at $t = 6$