INTEGRALS OF THE TYPE $\int f'(x)\,[f(x)]^n$

You have seen this type of integral before (e.g. with integrals like $\int 2x(x^2+1)^3\,dx = \frac{1}{4}(x^2+1)^4 + C$). This section will consider trigonometric integrals such as $\int \cos x \sin^2 x\,dx$, i.e. where $f(x) = \sin x$ and $f'(x) = \cos x$. In trigonometric integrals of this type the substitution will not always be given.

Example 16

(a) Find $\int \cos x \sin^2 x\,dx$. **(b)** Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$.

Solution

(a) $\int \cos x \sin^2 x\,dx$: Let $u = \sin x$, $\frac{du}{dx} = \cos x$

$$\begin{aligned}\int \cos x \sin^2 x\,dx &= \int u^2 \times \frac{du}{dx} \times dx \\ &= \int u^2\,du \\ &= \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C\end{aligned}$$

If you can recognise that $\int \cos x \sin^2 x\,dx$ is of the form $\int f'(x)(f(x))^2\,dx = \frac{1}{3}(f(x))^3 + C$, then you can write the answer immediately as $\int \cos x \sin^2 x\,dx = \frac{1}{3}\sin^3 x + C$.

(b) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$: Let $u = \cos x$, $\frac{du}{dx} = -\sin x$ For $x = \frac{\pi}{3}$, $u = \frac{1}{2}$; for $x = \frac{\pi}{2}$, $u = 0$

$$\begin{aligned}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x\,dx &= \int_{\frac{1}{2}}^{0} -u^2 \times \frac{du}{dx} \times du \\ &= \int_{\frac{1}{2}}^{0} -u^2\,du \\ &= \left[-\frac{1}{3}u^3\right]_{\frac{1}{2}}^{0} = 0 + \frac{1}{3} \times \left(\frac{1}{2}\right)^3 = \frac{1}{24}\end{aligned}$$

Note: The result $\int_{\frac{1}{2}}^{0} -u^2\,du = \int_{0}^{\frac{1}{2}} u^2\,du$ (reversing the limits of the integral and changing the sign of the integrand) could have been used. Also: $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x\,dx = -\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (-\sin x)\cos^2 x\,dx$

$$\begin{aligned}&= -\left[\frac{1}{3}\cos^3 x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= -\frac{1}{3}\left(0^3 - \left(\frac{1}{2}\right)^3\right) = \frac{1}{24}\end{aligned}$$

Substitution simplified

If you treat $\frac{du}{dx}$ like a fraction, then $\frac{du}{dx} \times dx = du$.

If $u = \sin x$, then $\frac{du}{dx} = \cos x$ can be written as $du = \cos x\,dx$.

You are really replacing $f'(x)\,dx$ by du. This makes the algebra involved much simpler.

Example 17

Find $\int \tan^2 x \sec^2 x\,dx$.

Solution

Let $u = \tan x$, $du = \sec^2 x\,dx$: $\int \tan^2 x \sec^2 x\,dx = \int u^2\,du$

$$= \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$$