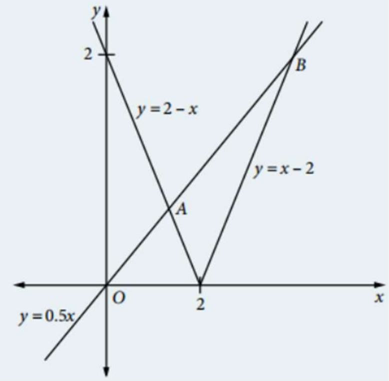
INEQUALITIES INVOLVING ABSOLUTE VALUE AND SQUARE ROOTS

Inequalities involving absolute values

Example 5

Solve $|x-2| < \frac{x}{2}$.



Solution

From the definition of absolute value, you know that $|x-2|$ is simply equal to $x-2$ where $x \geq 2$, but it is equal to $-(x-2)$ where $x < 2$.

Using graphical method (**Method 4** from Example 3):

Graph $y = |x-2|$ and $y = \frac{x}{2}$.

Note that the 'V'-shaped absolute value graph consists of the ray $y = 2 - x$ for $x < 2$ and the ray $y = x - 2$ for $x \geq 2$.

The solution of the inequality $|x-2| < \frac{x}{2}$ is shown on the graph where the 'V'-shaped absolute value graph is below the straight line $y = \frac{x}{2}$. You need to find the x-coordinates of the points of intersection A and B.

- At A, $y = 2 - x$ and $y = \frac{x}{2}$ intersect. Solving these: $x = \frac{4}{3}$
- At B, $y = x - 2$ and $y = \frac{x}{2}$ intersect. Solving these: $x = 4$

Thus the solution is: $\frac{4}{3} < x < 4$

Using analytical method (**Method 1** from Example 3):

For $x \geq 2$: $x - 2 < \frac{x}{2}$

$$\frac{x}{2} < 2$$

$$x < 4$$

Both $x \geq 2$ and $x < 4$ must be true.

$\therefore 2 \leq x < 4$

For $x < 2$: $2 - x < \frac{x}{2}$

$$2 < \frac{3x}{2}$$

$$x > \frac{4}{3}$$

Both $x < 2$ and $x > \frac{4}{3}$ must be true.

$\therefore \frac{4}{3} < x < 2$

Thus the complete solution is $\frac{4}{3} < x < 4$.

Example 6

Solve $|x-4|+|x+2|>7$.

Solution

Although it is possible to solve this analytically, a graphical approach is easier.

- First sketch $y=|x-4|$ and $y=|x+2|$ on the same axes, writing the equations of their rays.
  Remember that $y=|x-4|$ is made up of the ray $y=x-4$ (with positive gradient) and $y=4-x$.
  Similarly, $y=|x+2|$ is made up of $y=x+2$ and $y=-x-2$.

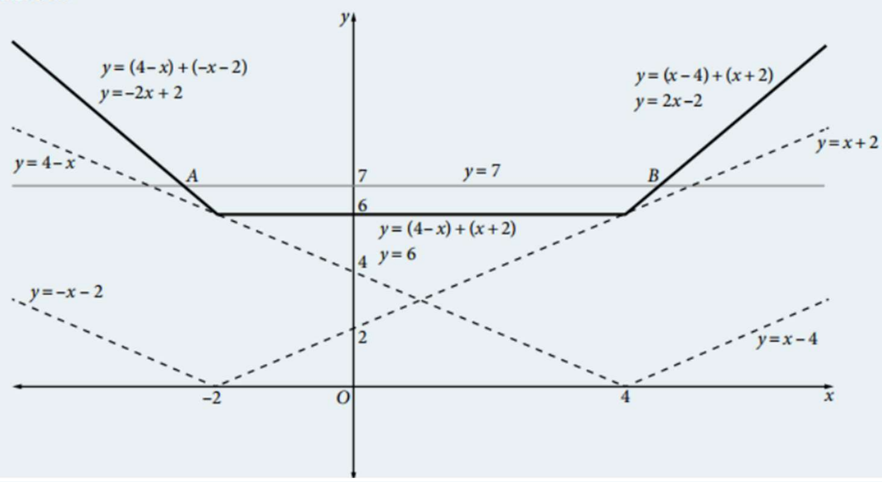
  Note that the graphs have the bases of their 'V'-shapes at $x=4$ and $x=-2$.

Next, sketch $y=|x-4|+|x+2|$ by determining the equations of each of its three parts:

- For $x<-2$, find the equation by adding the ordinates (y values) that apply. Over this domain the two rays that apply are $y=-x-2$ and $y=4-x$, so the equation is $y=(-x-2)+(4-x)$, i.e. $y=-2x+2$. This can easily be sketched. At $x=-2$, $y=-2(-2)+2=6$, so the ray starts at $(-2,6)$ and is drawn back to the left with a gradient of -2 (which is steeper than the existing rays on the diagram).
- For $-2<x<4$, the two rays that apply are $y=-x-2$ and $y=x-4$, so the equation is $y=(-x-2)+(x-4)$, i.e. $y=6$. Add this horizontal interval to the graph.
- For $x>4$, the two rays that apply are $y=x+2$ and $y=x-4$, so the equation is $y=(x+2)+(x-4)$, i.e. $y=2x-2$.
  Draw this ray on the graph. It starts at $(4,6)$ and proceeds to the right with a gradient of 2.

You have now sketched the 'trough'-shaped graph of $y=|x-4|+|x+2|$ and found the equations of each of its three parts.

- Draw the horizontal line $y=7$ on the graph and label points A and B where $y=|x-4|+|x+2|$ and $y=7$ intersect.



To solve $|x-4|+|x+2|>7$, you need to find the x values for which the graph is above the horizontal line $y=7$, which means you need to find the x-coordinates of A and B.

- To find A, solve simultaneously $y=-2x+2$ and $y=7$: $x=-\frac{5}{2}$
- To find B, solve simultaneously $y=2x-2$ and $y=7$: $x=\frac{9}{2}$

Hence the solution is $x<-\frac{5}{2}$, $x>\frac{9}{2}$.

Inequalities involving square roots

Example 7

(a) Sketch $y = \sqrt{1 - x^2}$ and state the domain of this function.

(b) On the same diagram sketch $y = |x - 1|$.

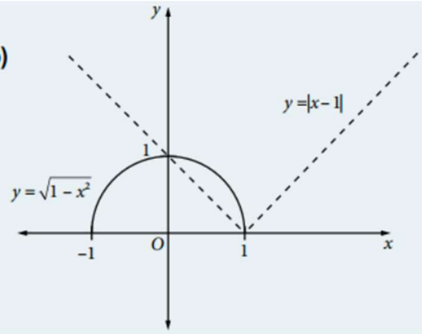
(c) Hence, or otherwise, solve $|x - 1| \geq \sqrt{1 - x^2}$.

(d) Solve $x - 1 \geq \sqrt{1 - x^2}$.

Solution

(a) The sketch is shown at right.
The domain is $-1 \leq x \leq 1$.

(b)



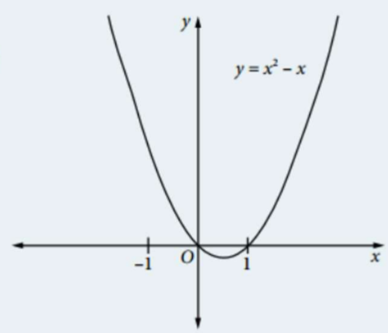
(c) To solve $|x - 1| \geq \sqrt{1 - x^2}$ you need to state the x values for which the 'V'-shaped absolute graph is on or above the semicircle. The graphs intersect at $x = 0$ and $x = 1$. Also remember the domain from **(a)**, i.e. that the only applicable x values are between -1 and 1. Thus the solution is $-1 \leq x \leq 0$, $x = 1$.

Alternatively:

In this inequality, both sides are known to be non-negative (as $\sqrt{1 - x^2}$ means the positive square root), so you can square both sides and know that the inequality will not change.

$$\left(|x - 1|\right)^2 \geq \left(\sqrt{1 - x^2}\right)^2$$
$$x^2 - 2x + 1 \geq 1 - x^2$$
$$2x^2 - 2x \geq 0$$
$$x^2 - x \geq 0$$

Graphing $y = x^2 - x$:
The parabola is on or above the x-axis for $x \leq 0$, $x \geq 1$.
But the domain is restricted to $-1 \leq x \leq 1$, so the solution is $-1 \leq x \leq 0$, $x = 1$.



(d) Graphing $y = \sqrt{1 - x^2}$ and $y = x - 1$:

The only point of intersection is at $x = 1$.

You now find the x values for which the straight line is on or above the semicircle, keeping in mind that x values must be between -1 and 1.

Thus the solution is $x = 1$.