SOLVING DIFFERENTIAL EQUATIONS OF THE FORM *dy/dx* = *f(x)*

1 Write the general solution of the following differential equations.

(a) $\frac{dy}{dx} = 2x - 1$ (b) $f'(x) = x^2\sqrt{x}$ (c) $y'(x) = 2\cos 2x$ (d) $y'(x) = 2\cos^2 x$

a) $\int (2x-1)dx = x^2 - x + C$ So the general solution is $y = x^2 - x + C$

b) $\int x^2\sqrt{x}\,dx = \int x^{5/2}\,dx = \frac{x^{5/2+1}}{\frac{5}{2}+1} + C = \frac{x^{7/2}}{7/2} + C = \frac{2}{7}x^{7/2} + C$

So the general solution of $f'(x) = x^2\sqrt{x}$ is $f(x) = \frac{2}{7}x^{7/2} + C$

c) $\int 2\cos 2x\,dx = 2\frac{\sin 2x}{2} + C = \sin 2x + C$

So the general solution of $y'(x) = 2\cos 2x$ is $y(x) = \sin 2x + C$

d) $\int 2\cos^2 x\,dx = \int (\cos 2x + 1)\,dx = \int \cos 2x + \int dx$

$= \frac{\sin 2x}{2} + x + C$

So the general solution of the differential equation $y'(x) = 2\cos^2 x$ is $y(x) = \frac{\sin 2x}{2} + x + C$

SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $dy/dx = f(x)$

1 Write the general solution of the following differential equations.

(e) $\frac{dz}{dt} = \frac{1}{t^2+4}$ (f) $\frac{dz}{dt} = \frac{t}{t^2+4}$ (g) $\frac{dx}{d\theta} = \sin^2\theta + \cos^2\theta$ (h) $f'(x) = 1 - e^{\frac{-x}{2}}$

e) $\int \frac{1}{t^2+4}\,dt = \int \frac{1}{2^2+t^2}\,dt = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + C$

So the general solution to $\frac{dz}{dt} = \frac{1}{t^2+4}$ is $z(t) = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + C$

f) $\int \frac{t}{t^2+4}\,dt = \frac{1}{2}\int \frac{2t}{t^2+4}\,dt = \frac{1}{2}\ln(t^2+4) + C$

So the general solution to $\frac{dz}{dt} = \frac{t}{t^2+4}$ is $z(t) = \ln\sqrt{t^2+4} + C$

g) $\frac{dx}{d\theta} = 1$ So the general solution is $x = \theta + C$

h) $\int (1 - e^{-x/2})\,dx = \int dx - \int e^{-x/2}\,dx = x - e^{-x/2} \times (-2) + C$

$ = x + 2e^{-x/2} + C$

So the general solution of the differential equation

$f'(x) = 1 - e^{-x/2}$ is $f(x) = x + 2e^{-x/2} + C$

SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $dy/dx = f(x)$

2 Find the particular solution of the following differential equations.

(a) $\frac{dy}{dx} = 2x^3 - x + 1$, given that $y = 2$ where $x = 1$ **(e)** $\frac{dx}{d\theta} = \frac{\sin\theta}{2+\cos\theta}$, given that $x = 1$ where $\theta = \pi$

a) $\int (2x^3 - x + 1)\,dx = 2\frac{x^4}{4} - \frac{x^2}{2} + x + C = \frac{x^4}{2} - \frac{x^2}{2} + x + C$

The general solution of the differential equation $\frac{dy}{dx} = 2x^3 - x + 1$

is $y(x) = \frac{x^4}{2} - \frac{x^2}{2} + x + C$

Now for the particular solution for which $y(1) = 2$

$y(1) = \frac{1^4}{2} - \frac{1^2}{2} + 1 + C = 1 + C$ But $y(1) = 2$ so $C = 1$

This particular solution is $y(x) = \frac{x^4}{2} - \frac{x^2}{2} + x + 1$

e) $\int \frac{\sin\theta}{2+\cos\theta}\,d\theta = -\ln(2+\cos\theta) + C$ which is the general solution to the diff. equ. $\frac{dx}{d\theta} = \frac{\sin\theta}{2+\cos\theta}$.

(*) NOTE.

Now for the particular solution for which $x = 1$ where $\theta = \pi$

$x(\theta) = -\ln(2+\cos\theta) + C$

$x(\pi) = -\ln(2+\cos\pi) + C = -\ln(2-1) + C = -\ln 1 + C = C$

So $C = 1$

This particular solution is $x(\theta) = -\ln(2+\cos\theta) + 1$

SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $dy/dx = f(x)$

3 Find the particular solution of the following differential equations.

(a) $\frac{dy}{dx} = \frac{1}{\sqrt{9-x^2}}$, $y(0) = 1$ (b) $\frac{dx}{dt} = \frac{t}{t^2+1}$, $x = 1$ where $t = 0$ (c) $\frac{dx}{dy} = \frac{y}{2y-2}$, given that $x = 1$ where $y = 2$

a) $\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$ So the general solution to the diff. eq. is $y(x) = \sin^{-1}\left(\frac{x}{3}\right) + C$

Then, for the particular solution,
$y(0) = \sin^{-1}\left(\frac{0}{3}\right) + C = C$ but $y(0) = 1$ $\therefore C = 1$

The particular solution is $y(x) = \sin^{-1}\left(\frac{x}{3}\right) + 1$

b) $\int \frac{t}{t^2+1} dt = \frac{1}{2}\int \frac{2t}{1+t^2} dt = \frac{1}{2}\ln(1+t^2) + C$

The general solution to the diff. eq. $\frac{dx}{dt} = \frac{t}{t^2+1}$ is
$x(t) = \frac{1}{2}\ln(1+t^2) + C$

For the particular solution, $x(0) = \frac{1}{2}\ln(1+0^2) + C = C$ so $C = 1$

the particular solution is $x(t) = \frac{1}{2}\ln(1+t^2) + 1 = \ln\sqrt{1+t^2} + 1$

c) $\int \frac{y}{2y-2} dy = \frac{1}{2}\int \frac{y}{y-1} dy = \frac{1}{2}\int \frac{y-1+1}{y-1} dy$

$= \frac{1}{2}\left[\int\left(1 + \frac{1}{y-1}\right) dy\right] = \frac{1}{2}\left[y + \ln|y-1|\right] + C$

So the general solution is $x(y) = \frac{y}{2} + \ln\sqrt{|y-1|} + C$

For the particular solution,
$x(2) = \frac{2}{2} + \ln\sqrt{|2-1|} + C = 1 + C$ But $x(2) = 1$
so $C = 0$ $x(y) = \frac{y}{2} + \ln\sqrt{|y-1|}$

SOLVING DIFFERENTIAL EQUATIONS OF THE FORM *dy/dx = f(x)*

5 (a) Show that $\frac{d}{dx}\left(xe^x\right) = e^x + xe^x$. (b) Hence find $\int xe^x\,dx$.

(c) Find the particular solution of the differential equation $\frac{dy}{dx} = xe^x$, given $y(0) = -1$.

(d) Find the particular solution of the differential equation $\frac{dy}{dx} = xe^x - e^x$, given $y(0) = -2$.

(e) Hence find the particular solution for the second-order differential equation $\frac{d^2y}{dx^2} = xe^x$, given that $\frac{dy}{dx} = -1$ and $y = -2$ where $x = 0$.

a) $\frac{d}{dx}\left(xe^x\right) = xe^x + e^x$ (product rule)

b) from a) $xe^x = \frac{d}{dx}\left(xe^x\right) - e^x$

$\therefore \int xe^x\,dx = \int\left[\frac{d}{dx}\left(xe^x\right) - e^x\right]dx = xe^x - \int e^x\,dx$

$= xe^x - e^x + C$

c) The general solution is $y(x) = xe^x - e^x + C$

then $y(0) = 0 - 1 + C = C - 1$ But $y(0) = -1$ so $C = 0$

This particular solution is $y(x) = xe^x - e^x$

d) $\int\left(xe^x - e^x\right)dx = \int xe^x\,dx - \int e^x\,dx = xe^x - e^x - e^x + C$

$= xe^x - 2e^x + C$ (General solution)

For the particular solution, $y(0) = -2 + C$ and $y(0) = -2$ so $C = 0$

$y(x) = xe^x - 2e^x$ is that particular solution

e) if $\frac{d^2y}{dx^2} = xe^x$ then $\frac{dy}{dx} = \int xe^x\,dx = xe^x - e^x + C$

and then $y = \int\left(xe^x - e^x + C\right)dx = \left[\int xe^x\,dx\right] - e^x + Cx$

$y(x) = xe^x - e^x + D - e^x + Cx = xe^x - 2e^x + D + Cx$

$y(0) = -2 + D = -2$ so $D = 0$

$\frac{d}{dx}y(0) = 0e^0 - e^0 + C = -1 + C$ and $-1 + C = -1$ so $C = 0$

$y(x) = xe^x - 2e^x = e^x(x-2)$ is that particular solution

SOLVING DIFFERENTIAL EQUATIONS OF THE FORM *dy/dx = f(x)*

7 **(a)** Show that $\frac{d}{dx}\left(x + x\tan^{-1}x - \frac{1}{2}\log_e\left(x^2+1\right)\right) = \tan^{-1}x + 1$.

(b) Using **(a)**, find the particular solution of the differential equation $\frac{dy}{dx} = \tan^{-1}x + 1$ if $y(0) = 0$.

a) $\frac{d}{dx}\left[x + x\tan^{-1}x - \frac{1}{2}\ln(x^2+1)\right] = 1 + \frac{d}{dx}\left(x\tan^{-1}x\right) - \frac{1}{2}\frac{d}{dx}\ln(x^2+1)$

$\text{———} = 1 + \left[\tan^{-1}x + x \times \frac{1}{1+x^2}\right] - \frac{1}{2} \times \frac{1}{x^2+1} \times 2x$

$\text{———} = 1 + \tan^{-1}x + \frac{x}{x^2+1} - \frac{x}{x^2+1}$

$\therefore \text{———} = 1 + \tan^{-1}x$

b) $\int(\tan^{-1}x + 1)\,dx = \int\frac{d}{dx}\left(x + x\tan^{-1}x - \frac{1}{2}\ln(x^2+1)\right)dx$

$\text{———} = x + x\tan^{-1}x - \frac{1}{2}\ln(x^2+1) + C$

which is the general solution of the diff. equa.

For the particular solution:

$y(0) = 0 + 0 \times \tan^{-1}0 - \frac{1}{2}\ln(0^2+1) + C = C$

so $C = 0$ as $y(0) = 0$

$y(x) = x + x\tan^{-1}x - \frac{1}{2}\ln(x^2+1)$ is the particular solution for which $y(0) = 0$

SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $dy/dx = f(x)$

8 (a) If $\frac{dy}{dx} = \frac{1}{2}\left(e^x - e^{-x}\right)$ with initial condition $y(0) = 1$, find y.

(b) If $\frac{dz}{dx} = \frac{1}{2}\left(e^x + e^{-x}\right)$ with initial condition $z(0) = 0$, find z.

(c) Hence show that if $\frac{d^2y}{dx^2} = \frac{1}{2}\left(e^x - e^{-x}\right)$ with $y(0) = 0$ and $y'(0) = 1$, then $y = \frac{1}{2}\left(e^x - e^{-x}\right)$ is a particular solution of this equation.

a) $\int \frac{1}{2}\left(e^x - e^{-x}\right) dx = \frac{1}{2}\left[\int \left(e^x - e^{-x}\right) dx\right] = \frac{1}{2}\left[e^x + e^{-x}\right] + C$ general solution.

$y(0) = \frac{1}{2}\left[e^0 + e^{-0}\right] + C = 1 + C$ so $C = 0$ as $y(0) = 1$

The particular solution is $y(x) = \frac{1}{2}\left[e^x + e^{-x}\right]$

Note: this function $y(x) = \frac{e^x + e^{-x}}{2}$ is named "hyperbolic cosine" noted $\cosh(x)$ - it can be calculated on the calculator using the $\boxed{hyp}$

b) $\int \frac{1}{2}\left(e^x + e^{-x}\right) dx = \frac{1}{2}\left[e^x - e^{-x}\right] + C$

$z(0) = \frac{1}{2}\left[1 - 1\right] + C = 0$ so $C = 0$

$z(x) = \frac{e^x - e^{-x}}{2}$ is named "hyperbolic sine", noted $\sinh(x)$

NOTE that: $[\cosh(x)]' = \sinh(x)$ and $[\sinh(x)]' = \cosh(x)$

c) if $y(x) = \sinh(x)$, then $y'(x) = \cosh x$ and $y''(x) = (\cosh(x))' = \sinh(x)$.

$\therefore y(x) = \sinh(x)$ is a solution to $y'' = y$

SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $dy/dx = f(x)$

10 An oil tanker hits a reef and spills oil into the sea. The oil spills from the tanker at a rate of $\frac{10^6 t}{t^4+16}$ litres/day, where t is the number of days since the tanker first hit the reef.

It is known that $\int \frac{t}{t^4+16}dt = \frac{1}{8}\arctan\left(\frac{t^2}{4}\right)+C$.

(a) If V litres is the volume of oil spilled into the sea in the first T days, find V in terms of T.

The local newspaper report stated, 'It is expected that eventually 300 000 litres of oil will spill into the sea.'

(b) Determine whether the newspaper report is in agreement with the model above.

a) $V = \int_0^T \frac{10^6 t}{t^4+16} dt = 10^6 \int_0^T \frac{t}{t^4+16} dt = 10^6 \times \frac{1}{8}\left[\tan^{-1}\left(\frac{t^2}{4}\right)\right]_0^T$

$V = \frac{10^6}{8}\left[\tan^{-1}\left(\frac{T^2}{4}\right) - \tan^{-1}(0)\right] = \frac{10^6}{8}\tan^{-1}\left(\frac{T^2}{4}\right)$

b) We need to find V when T gets very large, i.e.

$\lim_{T\to+\infty} \frac{10^6}{8}\left(\tan^{-1}\left(\frac{T^2}{4}\right)\right)$.

$\lim_{T\to+\infty} \frac{10^6}{8}\tan^{-1}\left(\frac{T^2}{4}\right) = \frac{10^6}{8} \times \underbrace{\lim_{T\to+\infty} \tan^{-1}\left(\frac{T^2}{4}\right)}_{\pi/2}$

$\text{———} = \frac{10^6}{8} \times \frac{\pi}{2}$

$\text{———} = \frac{\pi \times 10^6}{16} = 196,350$ Litres.

so closer to 200,000 Litres, not 300,000 Litres.