SOLVING EQUATIONS WITH LOGARITHMS

In the previous section you investigated the logarithm laws and the change of base rule. You will now see how to use these techniques in an algebraic setting to solve more difficult equations.

Note that when the notation 'log' is written without a base, then by convention you should assume it represents $\log_{10}$, the common logarithm.

Example 10

Solve the equations.

(a) $\log_{10} x = \log_{10} 9 + \log_{10} 3$ **(b)** $3\log_{10} x + 4 = 7\log_{10} x$

Solution

(a)

$$\log_{10} x = \log_{10} 9 + \log_{10} 3$$

Use $\log_a m + \log_a n = \log_a(mn)$: $\log_{10} x = \log_{10} 27$

If $\log a = \log b$ then $a = b$: $x = 27$

(b)

$$3\log_{10} x + 4 = 7\log_{10} x$$

Collect like terms: $4 = 4\log_{10} x$

Simplify: $\log_{10} x = 1$

If $\log_a n = y$ then $n = a^y$: $x = 10^1 = 10$

Example 11

For what value of x is $\log_2(x+1) - \log_2(x-1) = 3$ true?

Solution

For $\log x$ to exist, $x > 0$. Hence the equation requires $x + 1 > 0$ and so $x > -1$.

The equation also requires $x - 1 > 0$, and so $x > 1$.

Therefore, for a solution to exist for this equation it requires $x > 1$.

Use $\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)$: $\log_2\left(\frac{x+1}{x-1}\right) = 3$

If $\log_a n = y$ then $n = a^y$: $\frac{x+1}{x-1} = 2^3$

$$x + 1 = 8x - 8$$

Solve equation: $x = 1\frac{2}{7}$

As $1\frac{2}{7} > 1$, this is a valid solution to the equation.

Example 12

Solve, giving answers correct to 3 decimal places: **(a)** $2^x = 7$ **(b)** $3^{x+1} = 12$

Solution

(a)

$$2^x = 7$$

Take logs to base 10: $\log_{10} 2^x = \log_{10} 7$

$$x\log_{10} 2 = \log_{10} 7$$

$$x = \frac{\log_{10} 7}{\log_{10} 2}$$

$$x = 2.807 \quad \text{(3 d.p.)}$$

(b)

$$3^{x+1} = 12$$

Take logs to base 10: $\log_{10} 3^{(x+1)} = \log_{10} 12$

$$(x+1)\log_{10} 3 = \log_{10} 12$$

$$x + 1 = \frac{\log_{10} 12}{\log_{10} 3}$$

$$x = \frac{\log_{10} 12}{\log_{10} 3} - 1$$

$$x = 1.262 \quad \text{(3 d.p.)}$$

Example 13

Solve the inequalities: **(a)** $2^x > 9$ **(b)** $0.4^x < 0.3$

Solution

(a)

$$2^x > 9$$

$$x > \log_2 9$$

Change of base rule: $x > \dfrac{\log_{10} 9}{\log_{10} 2}$

$$x > 3.17 \quad \text{(2 d.p.)}$$

(b)

$$0.4^x < 0.3$$

$$x \log 0.4 < \log 0.3$$

$$x < \frac{\log_{10} 0.3}{\log_{10} 0.4}$$

Although this step looks correct, it contains an error. This is because $\log_{10} 0.4 < 0$, so you have divided by a negative quantity without reversing the direction of the inequality sign.

Correct result: $x > \dfrac{\log_{10} 0.3}{\log_{10} 0.4}$

$$x > 1.31 \quad \text{(2 d.p.)}$$

Be very careful when using logarithms with inequalities. Remember that if $0 < m < 1$ and $a > 1$, then $\log_a m < 0$, so when you divide by that logarithm you must reverse the direction of the inequality.

Example 14

How many years does it take for \$2000 to grow to \$3000, at 7% p.a. compound interest?

Note that 'p.a.' = 'per annum' = per year. The compound interest formula is $A = P\left(1 + \frac{r}{100}\right)^n$, where P is the initial money invested and A is the amount that P grows to after n periods of time (years) with interest applied at r% per period. Compound interest is investigated further in Chapter 18.

Solution

$A = 2000, r = 7, P = 3000.$ Find n: $2000 \times 1.07^n = 3000$

$$1.07^n = \frac{3}{2}$$

Take logs: $n = \log_{1.07} 1.5$ **OR** $n \log_{10} 1.07 = \log_{10} 1.5$

Use change of base rule **OR** rearrange: $n = \dfrac{\log_{10} 1.5}{\log_{10} 1.07}$

$$n \approx 5.993 \text{ years}$$

The value of n is just under 6 years. Assuming that interest is added at the end of each year, it will take 6 years.