DIFFERENTIAL CALCULUS - CHAPTER REVIEW

1 Differentiate:

(a) $\sin x + \tan 2x$ (b) $3\cos 4x - 5\sin 2x$ (c) $x\cos x$

(d) $\dfrac{\cos x}{\sin x}$ (e) $2e^{-x}\cos 3x$ (f) $\log_e(\sin 2x)$

a) $f(x) = \sin x + \tan 2x$ $\quad f'(x) = \cos x + \sec^2(2x) \times 2 = \cos x + 2\sec^2 2x$

b) $f(x) = 3\cos 4x - 5\sin 2x$ $\quad f'(x) = 12(-\sin 4x) - 10\cos 2x$

$f'(x) = -12\sin 4x - 10\cos 2x$

c) $f(x) = x\cos x$ $\quad u(x) = x$ $\quad u'(x) = 1$

$v(x) = \cos x$ $\quad v'(x) = -\sin x$

$f'(x) = 1 \times \cos x - x\sin x = \cos x - x\sin x$

d) $f(x) = \dfrac{\cos x}{\sin x} = \dfrac{u(x)}{v(x)}$ $\quad u(x) = \cos x$ $\quad u'(x) = -\sin x$

$v(x) = \sin x$ $\quad v'(x) = \cos x$

$f'(x) = \dfrac{-\sin x \times \sin x - \cos x \times \cos x}{\sin^2 x} = \dfrac{-1}{\sin^2 x} = (-1) \times \dfrac{1}{(\sin x)^2}$

$= -\csc^2 x$

e) $f(x) = 2e^{-x}\cos 3x = 2 \times u(x) \times v(x)$

$u(x) = e^{-x}$ $\quad u'(x) = -e^{-x}$

$v(x) = \cos 3x$ $\quad v'(x) = -3\sin 3x$

$f'(x) = 2\left[-e^{-x}\cos 3x - 3e^{-x}\sin 3x\right] = -2e^{-x}\left[\cos 3x + 3\sin 3x\right]$

f) $f(x) = \ln(\sin 2x) = g(h(x))$ $\quad$ with $g(x) = \ln x$ $\quad g'(x) = 1/x$

$h(x) = \sin 2x$ $\quad h'(x) = 2\cos 2x$

$f'(x) = \dfrac{1}{\sin 2x} \times 2\cos 2x = 2\cot(2x)$

2 Differentiate with respect to x:

(a) $(x^2+2x)e^x$ **(b)** $2e^{-x}\ln x$ **(c)** $\log_e(1+e^x)$

a) $u(x) = x^2 + 2x$ $v(x) = e^x$

$u'(x) = 2x + 2$ $v'(x) = e^x$

$f'(x) = (2x+2)e^x + e^x(x^2+2x) = e^x[x^2+4x+2]$

b) $f(x) = 2e^{-x}\ln x = 2\,u(x)\,v(x)$ $u(x) = e^{-x}$ $u'(x) = -e^{-x}$

$v(x) = \ln x$ $v'(x) = 1/x$

$f'(x) = 2[-e^{-x}\ln x + e^{-x} \times 1/x] = 2e^{-x}\left[\frac{1}{x} - \ln x\right]$

c) $f(x) = \ln[1+e^x]$ Chain Rule

$u(x) = \ln x$ $v(x) = 1 + e^x$

$u'(x) = 1/x$ $v'(x) = e^x$

$$f'(x) = \frac{1}{1+e^x} \times e^x = \frac{e^x}{1+e^x}$$

(d) $\log_e(x^2+2x)$ **(e)** $(x^2+3x)e^{-3x}$ **(f)** $e^{\sqrt{x}}+\log_e\sqrt{x}$

d) $f(x) = \ln\left[x^2+2x\right]$ chain rule

$$f'(x) = \frac{1}{x^2+2x} \times (2x+2) = \frac{2(x+1)}{x(x+2)}$$

e) $f(x) = (x^2+3x)\,e^{-3x}$ Product and chain ~~chain~~ Rule

$u(x) = x^2+3x$ $v(x) = e^{-3x}$

$u'(x) = 2x+3$ $v'(x) = -3\,e^{-3x}$

$$f'(x) = (2x+3)\,e^{-3x} - 3e^{-3x}(x^2+3x)$$

$$f'(x) = e^{-3x}\left[-3x^2-7x+3\right]$$

f) $f(x) = e^{\sqrt{x}} + \ln\sqrt{x} = e^{\sqrt{x}} + \frac{1}{2}\ln x$

First we differentiate $g(x) = e^{\sqrt{x}}$ (chain rule)

$u(x) = e^{x}$ $v(x) = \sqrt{x}$

$u'(x) = e^{x}$ $v'(x) = \frac{1}{2}x^{-1/2} = 1/2\sqrt{x}$

So $g'(x) = e^{\sqrt{x}} \times \dfrac{1}{2\sqrt{x}}$

$$\therefore\ f'(x) = \frac{e^{\sqrt{x}}}{2\sqrt{x}} + \frac{1}{2x}$$

3 The position x of a particle moving along a straight line at any time t is given by $x = 3 + 6\cos\frac{\pi t}{6}$.

(a) Find the position of the particle for values of $t = 0, 2, 4, 6, 8, 10, 12$.

(b) Find the velocity and acceleration of the particle when it first reaches the position $x = 0$.

a) $t=0 \quad x(0) = 3+6\cos 0 = 9$

$t=2 \quad x(2) = 3+6\cos\left(\frac{2\pi}{6}\right) = 3+6\cos\left(\frac{\pi}{3}\right) = 3+6\times\frac{1}{2} = 3+3 = 6$

$t=4 \quad x(4) = 3+6\cos\left(\frac{4\pi}{6}\right) = 3+6\cos\left(\frac{2\pi}{3}\right) = 3+6\times\left(\frac{-1}{2}\right) = 3-3 = 0$

$t=6 \quad x(6) = 3+6\cos\left(\frac{6\pi}{6}\right) = 3+6\cos(\pi) = 3-6 = -3$

$t=8 \quad x(8) = 3+6\cos\left(\frac{8\pi}{6}\right) = 3+6\cos\left(\frac{4\pi}{3}\right) = 3+6\times\left(\frac{-1}{2}\right) = 3-3 = 0$

$t=10 \quad x(10) = 3+6\cos\left(\frac{10\pi}{6}\right) = 3+6\cos\left(\frac{5\pi}{3}\right) = 3+6\times\frac{1}{2} = 3+3 = 6$

$t=12 \quad x(12) = 3+6\cos\left(\frac{12\pi}{6}\right) = 3+6\cos 2\pi = 3+6 = 9$

b) $x(t) = 3+6\cos\left(\frac{\pi t}{6}\right)$

$$\dot{x} = 6\left(-\sin\left(\frac{\pi t}{6}\right)\times\frac{\pi}{6}\right) = -\pi\sin\left(\frac{\pi t}{6}\right)$$

$$\ddot{x} = -\pi\times\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi t}{6}\right) = -\frac{\pi^2}{6}\cos\left(\frac{\pi t}{6}\right)$$

so for $t=4$ (i.e when $x=0$), we obtain

$$\dot{x}(4) = -\pi\sin\left(\frac{\pi\times 4}{6}\right) = -\pi\sin\left(\frac{2\pi}{3}\right) = -\pi\times\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}\pi}{2}$$

$$\ddot{x}(4) = -\frac{\pi^2}{6}\cos\left(\frac{\pi\times 4}{6}\right) = -\frac{\pi^2}{6}\cos\left(\frac{2\pi}{3}\right) = -\frac{\pi^2}{6}\left(-\frac{1}{2}\right) = \frac{\pi^2}{12}$$

6 Differentiate:

(a) $\log_e(x\tan x)$ **(b)** $\log_e\left(\dfrac{x^3-6}{e^{-x}-1}\right)$ **(c)** $\log_e = \left(\dfrac{\sqrt{x}\cos x}{1-\sin^2 x}\right)$

a) $f(x) = \ln(x\tan x) = g[h(x)]$ $\quad g(x) = \ln x \quad g'(x) = 1/x$

$h(x) = x\tan x$

to calculate h', we use the product rule.

$u(x) = x \qquad v(x) = \tan x$

$u'(x) = 1 \qquad v'(x) = \sec^2 x$

$h'(x) = \tan x + x\sec^2 x$

So $f'(x) = \dfrac{1}{x\tan x} \times [\tan x + x\sec^2 x] = \dfrac{1}{x} + \dfrac{\sec^2 x}{\tan x}$

$f'(x) = \dfrac{1}{x} + \dfrac{1}{\cos^2 x} \times \dfrac{1}{\frac{\sin x}{\cos x}} = \dfrac{1}{x} + \dfrac{1}{\sin x\cos x}$

b) $f(x) = \ln\left[\dfrac{x^3-6}{e^{-x}-1}\right] = \ln[x^3-6] - \ln[e^{-x}-1]$

$f'(x) = \dfrac{1}{x^3-6} \times (3x^2) - \dfrac{1}{e^{-x}-1} \times (-e^{-x}) = \dfrac{3x^2}{x^3-6} + \dfrac{e^{-x}}{e^{-x}-1}$

c) $f(x) = \ln\left[\dfrac{\sqrt{x}\cos x}{1-\sin^2 x}\right] = \ln\left[\dfrac{\sqrt{x}\cos x}{\cos^2 x}\right] = \ln\left[\dfrac{\sqrt{x}}{\cos x}\right] = \ln\sqrt{x} - \ln(\cos x)$
$= \dfrac{1}{2}\ln x - \ln(\cos x)$

$f'(x) = \dfrac{1}{2x} - \dfrac{1}{\cos x} \times (-\sin x) = \dfrac{1}{2x} + \tan x$

7 Differentiate:

(a) $x^3\,10^x$ **(b)** $\sin x + \log_a x$ **(c)** $2^x + 3^x + 4^x$ **(d)** $\dfrac{a^x}{\log_a x}$

a) $f(x) = x^3\,10^x = x^3\,e^{x\ln 10}$

$u(x) = x^3$ $v(x) = e^{x\ln 10} = 10^x$

$u'(x) = 3x^2$ $v'(x) = e^{x\ln 10} \times \ln 10 = 10^x \ln 10$

So $f'(x) = 3x^2\,10^x + x^3\,10^x \ln 10 = x^2 10^x\left[3 + x\ln 10\right]$

b) $f(x) = \sin x + \log_a x = \sin x + \dfrac{\ln x}{\ln a}$

$f'(x) = \cos x + \dfrac{1}{x\ln a}$

c) $f(x) = 2^x + 3^x + 4^x = e^{x\ln 2} + e^{x\ln 3} + e^{x\ln 4}$

$f'(x) = \ln 2\,e^{x\ln 2} + \ln 3\,e^{x\ln 3} + \ln 4\,e^{x\ln 4}$

$f'(x) = \ln 2 \times 2^x + \ln 3 \times 3^x + \ln 4\,4^x$

d) $f(x) = \dfrac{a^x}{\log_a x} = \dfrac{e^{x\ln a}}{\dfrac{\ln x}{\ln a}} = \ln a \times \dfrac{e^{x\ln a}}{\ln x} = \ln a\,\dfrac{u(x)}{v(x)}$

$u(x) = e^{x\ln a} = a^x$ $v(x) = \ln x$

$u'(x) = \ln a\,e^{x\ln a} = \ln a\,a^x$ $v'(x) = 1/x$

$f'(x) = \ln a\left[\dfrac{\ln a\,a^x \times \ln x - a^x/x}{\ln^2 x}\right]$

$f'(x) = a^x \ln a\left[\dfrac{\ln a \ln x - 1/x}{\ln^2 x}\right]$

$f'(x) = a^x \ln a\left[\dfrac{\ln a}{\ln x} - \dfrac{1}{x\ln^2 x}\right] = a^x \ln a\left[\dfrac{1}{\log_a x} - \dfrac{1}{x\ln^2 x}\right]$