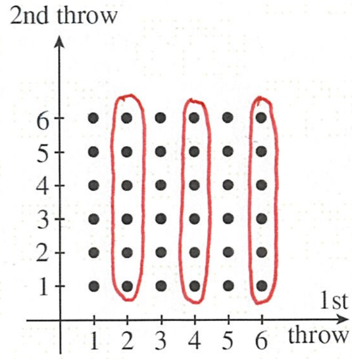
CONDITIONAL PROBABILITY (from CAMBRIDGE)

1 Two dice are thrown. The sample space of this experiment is shown in the dot diagram alongside. Following the throw, it is revealed that the first die shows an even number.



- a Copy the diagram and circle the reduced sample space.
- b Find the conditional probability of getting two sixes.
- c Find the conditional probability of getting at least one six.
- d Find the conditional probability that the sum of the two numbers is five.

b) Probability (two sixes) $= \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$

c) P(at least one six) = P(2 sixes) + P(2 and 6) + P(4 and 6) + P(6 and other)

$= \frac{1}{18} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{5}{6}$

$= 4/9$

d) P(sum is five) = P(2 and 3) + P(4+1) $= \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{6} = \frac{2}{18} = \frac{1}{9}$

2 A poll is taken amongst 1000 people to determine their voting patterns in the last election.

	Coalition	Labor	Other	Total
Male	130	340	110	580
Female	210	190	20	420
Total	340	530	130	1000

- a Determine the probability that a particular person in the group voted Coalition.
- b What is the probability that a particular person voted Labor, given that they were female?
- c What is the probability that a particular member of the group was male, if it is known that they voted Coalition?
- d Robin voted neither Coalition nor Labor. What is the probability that Robin was female?

a) P(person voted Coalition) $= 340/1000 = 0.34$

b) P(person voted Labor | they're female) $= 190/420 = \frac{19}{42} \approx 0.45$

c) P(person male | they voted Coalition) $= \frac{130}{340} = \frac{13}{34} \approx 0.38$

d) P(Robin female | Robin voted neither Coal or Lab) $= \frac{20}{130} = \frac{2}{13}$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

3 A student is investigating if there is any relationship between those who choose Mathematics Extension 1 (M) and those who choose English Extension 1 (E) at his school.

	M	$\overline{M}$	Total
E	29	27	56
$\overline{E}$	95	42	137
Total	124	69	193

a Copy and complete the table by filling in the totals.

b Find the probability that a particular student chose:

i neither Maths Extension 1 nor English Extension 1?

ii English Extension 1, given that they chose Maths Extension 1?

iii Maths Extension 1, if it is known that they chose English Extension 1?

iv Maths Extension 1, given that they did not choose English Extension 1?

b) i) $P(\text{neither Maths or English}) = \dfrac{42}{193} \simeq 0.22$

ii) $P(\text{English} \mid \text{they chose Maths}) = \dfrac{29}{124} \simeq 0.23$

iii) $P(\text{Maths} \mid \text{they chose English}) = \dfrac{29}{56} \simeq 0.52$

iv) $P(\text{Maths} \mid \text{they didn't chose English}) = \dfrac{95}{137} \simeq 0.69$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

4 Two cards are drawn from a standard pack — the first card is replaced and the pack shuffled before the second card is drawn — and the suit of each card (S, H, D or C) is noted by the game master. A player wants to know the probability that both cards are hearts.

a What is the probability if nothing else is known?

b The first card is known to be a heart. List the reduced sample space. What is the conditional probability of two hearts?

c List the reduced sample space if at least one of the cards is known to be a heart. What is the probability of two hearts?

d List the reduced sample space if the first card is known to be red. What is the probability of two hearts?

a) $P(\text{both cards are hearts}) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$

b) $P(\text{two hearts} \mid \text{the 1st card is a heart}) = \frac{1}{4}$

HH, HD, HC, HS

c) HH, HD, HC, HS, DH, CH, SH

$P(\text{two hearts}) = \frac{1}{7}$

d) HH, HD, HC, HS, DH, DD, DC, DS

$P(\text{two hearts}) = \frac{1}{8}$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

5 In a certain game, the player tosses two coins and then throws a die. The sample space is shown in the table.

The rules of the game assign one point for each head, and zero points for each tail. This score is then added to the score on the die.

	1	2	3	4	5	6
HH	3	4	5	6	7	8
HT	2	3	4	5	6	7
TH	2	3	4	5	6	7
TT	1	2	3	4	5	6

a Copy the table and fill in the point score for each outcome.
b Find the probability that a player gets more than 7 points.
c Suppose it is known that he has thrown two heads. Find the probability that a player gets more than 7 points.
d Find the probability that a player got an odd number of heads, given that their score was odd.

b) P(more than 7 points) = P(score is 8) = $1/24$

c) P(score greater than 7 | he has thrown 2 Heads) = $\frac{1}{6}$

d) P(odd number of heads | score is odd) =

$$= \frac{P(\text{odd number of heads AND score is odd})}{P(\text{score is odd})}$$

$$= \frac{6/24}{12/24}$$

$$= \frac{6}{12} = \frac{1}{2}$$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

6 Use the formula $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ to answer the following questions.

a Find $P(A|B)$ if $P(A \cap B) = 0.5$ and $P(B) = 0.7$.

b Find $P(A|B)$ if $P(A \cap B) = 0.15$ and $P(B) = 0.4$.

c Find $P(E|F)$ if $P(E \cap F) = 0.8$ and $P(F) = 0.95$.

a) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.5}{0.7} = \dfrac{5}{7}$

b) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.15}{0.4} = \dfrac{3}{8}$

c) $P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{0.8}{0.95} = \dfrac{16}{19}$

7 The two events A and B in the following experiments are known to be independent.

a $P(A) = 0.4$, and $P(B) = 0.6$. Find $P(A \cap B)$.

b $P(A) = 0.3$, and $P(B) = 0.5$. Find $P(A \cap B)$.

c $P(A) = 0.4$, and $P(B) = 0.6$. Find $P(A|B)$.

d $P(A) = 0.7$, and $P(B) = 0.2$. Find $P(A|B)$.

a) $P(A \cap B) = P(A) \times P(B) = 0.4 \times 0.6 = 0.24$

b) $P(A \cap B) = P(A) \times P(B) = 0.3 \times 0.5 = 0.15$

c) $P(A|B) = P(A) = 0.4$

d) $P(A|B) = P(A) = 0.7$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

8 Each of the following experiments involves two events, A and B. State in each case whether they are dependent or independent.

a $P(A|B) = 0.5$ and $P(A) = 0.4$ and $P(B) = 0.5$

b $P(A|B) = 0.3$ and $P(A) = 0.3$ and $P(B) = 0.6$

c $P(A|B) = \frac{3}{4}$ and $P(A) = \frac{2}{5}$ and $P(B) = \frac{3}{10}$

d $P(A) = 0.3$ and $P(B) = 0.7$ and $P(A \cap B) = 0.21$

e $P(A) = 0.2$ and $P(B) = 0.4$ and $P(A \cap B) = 0.8$

f $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{3}$

a) $P(A) = 0.4$ and $P(A|B) = 0.5$ so dependent

b) $P(A) = 0.3 = P(A|B)$ so independent

c) $P(A) = \frac{2}{5}$ and $P(A|B) = \frac{3}{4}$ so dependent

d) $P(A) \times P(B) = 0.3 \times 0.7 = 0.21 = P(A \cap B)$
so independent

e) We note that $P(A \cap B) > P(A)$
which is impossible.

f) $P(A \cap B) = \frac{1}{3}$ whereas $P(A) \times P(B) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

so independent

CONDITIONAL PROBABILITY (from CAMBRIDGE)

9 **a** Draw a table showing the sample space if two dice are thrown in turn and their sum is recorded.

b Highlight the reduced sample space if the sum of the two dice is 5. Given that the sum of the two dice is 5, find the probability that:

- **i** the first die shows a 1,
- **ii** at least one dice shows a 1,
- **iii** at least one of the dice shows an odd number.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

b) i) $P(\text{1st die is 1} \mid \text{sum of two dices is 5}) = \dfrac{P(\text{1st die is 1 AND sum is 5})}{P(\text{sum is 5})}$

$= \dfrac{1/36}{4/36} = \dfrac{1}{4}$

ii) $P(\text{at least one dice shows 1} \mid \text{sum of dices is 5}) = \dfrac{2}{4} = \dfrac{1}{2}$

iii) $P(\text{at least one dice shows an odd number} \mid \text{sum of dices is 5})$

$= \dfrac{4}{4} = 1$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

10 a For two events A and B it is known that $P(A \cup B) = 0.6$ and $P(A) = 0.4$ and $P(B) = 0.3$.

- **i** Use the addition formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to find $P(A \cap B)$.
- **ii** Use the formula for conditional probability to find $P(A|B)$.
- **iii** Find $P(B|A)$.

b Suppose that $P(V \cup W) = 0.7$ and $P(V) = 0.5$ and $P(W) = 0.35$. Find $P(V|W)$.

c Find $P(X|Y)$ if $P(X \cap Y) = 0.2$ and $P(X) = 0.3$ and $P(Y) = 0.4$.

d Find $P(A|B)$ if $P(A \cup B) = \frac{1}{3}$ and $P(A) = \frac{1}{5}$ and $P(B) = \frac{3}{10}$.

a) i) $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.3 - 0.6 = 0.1$

ii) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.1}{0.3} = \dfrac{1}{3}$

iii) $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.1}{0.4} = \dfrac{1}{4}$

b) $P(V|W) = \dfrac{P(V \cap W)}{P(W)} = \dfrac{P(V) + P(W) - P(V \cup W)}{P(W)}$

$ = \dfrac{0.5 + 0.35 - 0.7}{0.35} = \dfrac{0.15}{0.35} = \dfrac{3}{7}$

c) $P(X|Y) = \dfrac{P(X \cap Y)}{P(Y)} = \dfrac{0.2}{0.4} = \dfrac{1}{2}$

d) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A) + P(B) - P(A \cup B)}{P(B)}$

$ = \dfrac{\frac{1}{5} + \frac{3}{10} - \frac{1}{3}}{\frac{3}{10}}$

$ = \dfrac{1/6}{3/10} = \dfrac{5}{9}$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

11 Two dice are rolled. A three appears on at least one of the dice. Find the probability that the sum of the uppermost faces is greater than seven.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

P(sum greater than 7 | a 3 appears on at least one dice)

$$= \frac{P(\text{sum greater than 7 AND a 3 appears on at least one dice})}{P(\text{3 appears on at least one dice})}$$

$$= \frac{P(\text{3 then 5}) + P(\text{3 then 6}) + P(\text{5 then 3}) + P(\text{6 then 3})}{1 - P(\text{no 3 appears on any dice})}$$

$$= \frac{\frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}}{1 - \frac{25}{36}}$$

$$= \frac{\frac{4}{36}}{\frac{11}{36}} = \frac{4}{11}$$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

12 The members of a cricket team know that in half of the games they played last season, they won the game and their star player Arnav was playing. Arnav consistently plays in 80% of the games. What is the probability that they will win this Saturday if Arnav is playing?

$$P(\text{they will win the game} \mid \text{Arnav is playing})$$

$$= \frac{P(\text{they will win the game AND Arnav is playing})}{P(\text{Arnav is playing})}$$

$$= \frac{0.5}{0.8}$$

$$= \frac{5}{8} = 0.625$$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

13 a A couple has two children, the older of which is a boy. What is the probability that they have two boys?

b A couple has two children and at least one is a boy. What is the probability that they have two boys?

a) $P(\text{two boys}) = \frac{1}{2}$

b) $P(\text{two boys} \mid \text{at least one is a boy})$

$$= \frac{P(\text{two boys AND at least one is a boy})}{P(\text{at least one is a boy})}$$

$$= \frac{P(\text{two boys AND at least one is a boy})}{1 - P(\text{no boys})}$$

$$= \frac{\frac{1}{4}}{1 - \frac{1}{4}}$$

$$= \frac{1/4}{3/4} = \frac{1}{3}$$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

14 A couple has three children, each being either a boy or girl.

- **a** List the sample space.
- **b** Given that at least one is a boy, what is the probability that the oldest is male?
- **c** Given that at least one of the first two children is a boy, what is the probability that the oldest is male?

a) BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG

b) P(oldest is boy | at least one child is a boy)

$$= \frac{P(\text{oldest is boy AND at least one child is a boy})}{P(\text{at least one child is a boy})}$$

$$= \frac{4/8}{7/8} = \frac{4}{7}$$

c) P(oldest is boy | at least one of first two children is boy)

$$= \frac{P(\text{oldest is boy AND at least one of first two is a boy})}{P(\text{at least one of first two is boy})}$$

$$= \frac{4/8}{6/8} = \frac{4}{6} = \frac{2}{3}$$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

15 A card is drawn from a standard pack. The dealer tells the players that it is a court card (jack, queen or king).

- **a** What is the probability that it is a jack?
- **b** What is the probability that it is either a jack or a red card?
- **c** What is the probability that the next card drawn is a jack? Assume that the first card was not replaced.

a) $\frac{1}{3}$

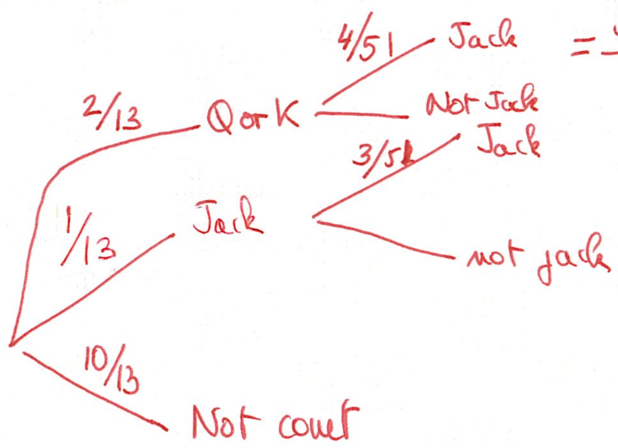
b) P(either jack OR red card | it's a court card)

= P(jack | court card) + P(queen heart or diamond | court card) + P(king heart or diamond | court card)

$= \frac{1}{3} + \frac{2}{12} + \frac{2}{12} = \frac{2}{3}$

c) P(next card is a jack | first card is a court card)

$$= \frac{P(\text{next card is a jack AND 1st card is a court card})}{P(\text{first card is a court card})}$$



$$= \frac{\frac{4}{51}\times\frac{2}{13}+\frac{3}{51}\times\frac{1}{13}}{\frac{3}{13}} = \frac{\frac{11}{663}}{\frac{3}{13}}$$

$$= \frac{11}{153}$$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

16 Two dice are tossed in turn and the outcomes recorded. Let A be the probability that the first die is odd. Let S be the probability that the sum is odd. Let M be the probability that the product is odd.

a Use the definition of independence in Box 20 to find which of the three pairs of events are independent.

b Confirm your conclusions using the product-rule test for independence in Box 21.

	1	2	3	4	5	6
1	2_{1}	3_{2}	4_{3}	5_{4}	6_{5}	7_{6}
2	3_{2}	4_{4}	5_{6}	6_{8}	7_{10}	8_{12}
3	4_{3}	5_{6}	6_{9}	7_{12}	8_{15}	9_{18}
4	5_{4}	6_{8}	7_{12}	8_{16}	9_{20}	10_{24}
5	6_{5}	7_{10}	8_{15}	9_{20}	10_{25}	11_{30}
6	7_{6}	8_{12}	9_{18}	10_{24}	11_{30}	12_{36}

$A = P(\text{1st die is odd}) = 1/3$

$S = P(\text{Sum is odd}) = \dfrac{2+4+6+4+2}{36} = \dfrac{18}{36} = \dfrac{1}{2}$

$M = P(\text{Product is odd}) = \dfrac{1+2+2+1+2+1}{36} = \dfrac{9}{36} = \dfrac{1}{4}$

$P(\text{1st die is odd AND Sum is odd}) = \dfrac{9}{36} = \dfrac{1}{4} \neq A \times S$

so not independent

$P(\text{1st die is odd AND Product is odd}) = \dfrac{3+3+3}{36} = \dfrac{9}{36} = \dfrac{1}{4} \neq A \times M$

so not independent

$P(\text{Sum is odd AND Product is odd}) = 0 \neq S \times M$

so not independent

CONDITIONAL PROBABILITY (from CAMBRIDGE)

17 The two events A and B in the following experiments are known to be independent.

a $P(A) = 0.4$ and $P(B) = 0.6$. Find $P(A \cup B)$.

b The probability of event A occurring is 0.6 and the probability of event B occuring is 0.3. What is the probability that either A or B occurs?

a) From $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$\Leftrightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$

But $P(A \cap B) = P(A) \times P(B)$ as A and B are independent

So $P(A \cup B) = 0.4 + 0.6 - 0.4 \times 0.6 = 0.76$

b) $P(A \cup B) = P(A) + P(B) - P(A) \times P(B)$

$\_\_\_\_ = 0.6 + 0.3 - 0.6 \times 0.3$

$\_\_\_\_ = 0.72$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

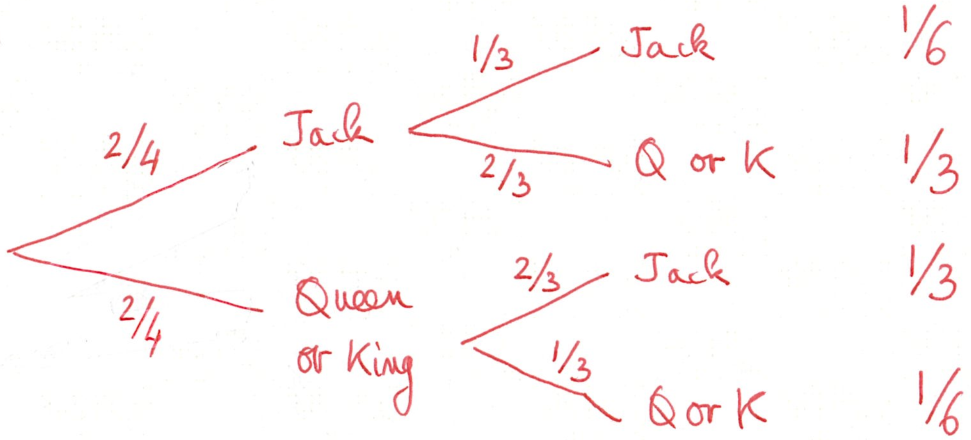
18 A set of four cards contains two jacks, a queen and a king. Bob selects one card and then, without replacing it, selects another. Find the probability that:

a both Bob's cards are jacks,

b at least one of Bob's cards is a jack

c given that one of Bob's cards is a jack, the other is also a jack.

a)



$$P(\text{both cards are Jacks}) = \frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$$

b) $$P(\text{at least one card is a Jack}) = \frac{1}{6} + 2 \times \frac{1}{3} = \frac{5}{6}$$

c) $$P(\text{other card is a Jack} \mid \text{one card is a Jack})$$

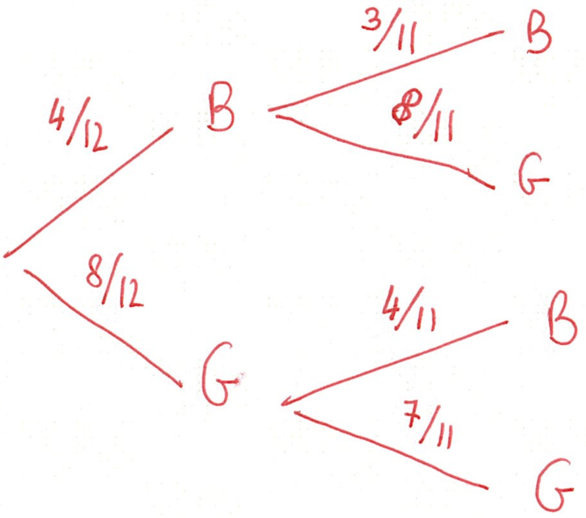
$$= \frac{P(\text{other card is a Jack AND one card is Jack})}{P(\text{one card is a Jack})}$$

$$= \frac{P(\text{both cards are Jacks})}{5/6} = \frac{1/6}{5/6} = \frac{1}{5}$$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

19 A small committee of two is formed from a group of 4 boys and 8 girls. If at least one of the members of the committee is a girl, what is the probability that both members are girls?

P(both members are girls | at least one of the member is a girl)



$$\frac{4\times 8}{12\times 11} = \frac{32}{132}$$

$$\frac{32}{132}$$

$$\frac{56}{132}$$

$$= \frac{\frac{56}{132}}{\frac{32}{132}+\frac{32}{132}+\frac{56}{132}} = \frac{56}{120} = \frac{7}{15}$$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

20 Jack and Ben have been tracking their success rate of converting goals in their rugby games. Jack converts 70% of his goals and Ben converts 60%. At a recent home game, both get a kick, but only one converts his goal. What is the probability that it was Ben?

P(scorer was Ben | both got a kick and only one scored)

$$= \frac{P(\text{scorer was Ben AND both got a kick and only one scored})}{P(\text{both got a kick AND only one scored})}$$

$$= \frac{0.6 \times (1-0.7)}{0.6 \times (1-0.7) + 0.7 \times (1-0.6)}$$

$$= \frac{0.6 \times 0.3}{0.6 \times 0.3 + 0.7 \times 0.4}$$

$$= \frac{0.18}{0.46} = \frac{9}{23}$$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

21 Susan picks two notes at random from four \$5, three \$10 and two \$20 notes. Given that at least one of the notes was \$10, what is the probability that Susan has picked up a total of \$20 or more?

4 x \$5

3 x \$10

2 x \$20

$$P(\text{Susan picks 2 notes totalling \$20 or more} \mid \text{at least one is \$10})$$

$$= \frac{P(\text{Susan picks 2 notes totalling at least \$20 AND at least one is \$10})}{P(\text{at least one note is a \$10})}$$

$$= \frac{P(10+10) + P(10+20) + P(20+10)}{1 - P(\text{no note is a \$10})}$$

$$= \frac{\frac{3}{9} \times \frac{2}{8} + \frac{3}{9} \times \frac{2}{8} + \frac{2}{9} \times \frac{3}{8}}{1 - \frac{6}{9} \times \frac{5}{8}}$$

$$= \frac{\frac{18}{72}}{\frac{42}{72}} = \frac{18}{42} = \frac{3}{7}$$

CONDITIONAL PROBABILITY (from CAMBRIDGE)

22 Researchers are investigating a potential new test for a disease, because although the usual test is totally reliable, it is painful and expensive.

The new test is intended to show a positive result when the disease is present. Unfortunately the test may show a *false positive*, meaning that the test result is positive, even though the disease is not present. The test may also show a *false negative*, meaning that the test result is negative even though the disease is in fact present.

The researchers tested a large sample of people who had symptoms that vaguely suggested that it was worth testing for the disease. They used the new test, and checked afterwards for the disease with the old reliable test, and came up with the following results:

- Of this sample group, 1% had the disease.
- Of those who had the disease, 80% tested positive and 20% tested negative.
- Of those who did not have the disease, 5% tested positive and 95% tested negative.

a What percentage of this group would test positive to this new test?

b Use the conditional probability formula $P(A \cap B) = P(A|B)P(B)$ to find the probability that a person tests positive, but does not have the disease.

c Find the probability that the new test gives a false positive. That is, find the probability the patient does not have the disease, given that the patient has tested positive.

d Find the probability that the new test gives a false negative. That is, find the probability the patient has the disease, given that the patient has tested negative.

e Comment on the usefulness of the new test.

a) P(new test positive) = P(person sick and test positive) + P(person healthy and test positive)
$= 0.01 \times 0.8 + 0.99 \times 0.05 = 0.0575$

b) P(new test positive AND person healthy) = P(test positive | person healthy) × P(healthy)
$= 0.05 \times 0.99 = 0.0495 = 4.95\%$

c) P(patient healthy | test positive) = $\dfrac{\text{P(patient healthy AND test positive)}}{\text{P(test positive)}}$
$= \dfrac{0.05}{0.0575} = 0.8695.. \simeq 87\%$

d) P(patient sick | test negative) = $\dfrac{\text{P(patient sick AND test negative)}}{\text{P(test negative)}}$
$= \dfrac{0.2}{1 - 0.0575} = 0.21... \approx 21\%$

e) It's most important that the number of false negatives is low, i.e. that almost all patients with the disease are picked up. False positives are scary for the patient, but further tests should determine that they don't have the disease.