WORKING WITH FUNCTIONS

Sum and difference of functions

The sum or difference of two functions is found by adding or subtracting any corresponding terms of the two functions, where this is possible. Where one of the functions has a restrictive domain, this domain must apply for the sum or difference function.

For a sum $h(x) = f(x) + g(x)$ or for a difference $k(x) = f(x) - g(x)$, any simplification of the resulting functions depends on the original functions.

Example 25

If $f(x) = x - 5$, $g(x) = x^2 + 3$, $h(x) = \sqrt{x+4}$ and $k(x) = x^3 - 2x^2 + 6$, find expressions for each of the following functions, stating the domain and range in each case.

(a) $f(x) + g(x)$ **(b)** $f(x) - g(x)$ **(c)** $f(x) + h(x)$ **(d)** $k(x) - g(x)$ **(e)** $g(x) - h(x)$

Solution

(a) $f(x) + g(x) = x - 5 + x^2 + 3$

$= x^2 + x - 2$

Domain is the set of real numbers.

For the range you need to find the least value of the quadratic expression.

Solve: $x^2 + x - 2 = 0$

$(x - 1)(x + 2) = 0$

so $x = -2$ or 1

Least value occurs when: $x = \frac{-2+1}{2} = -\frac{1}{2}$

$x = -\frac{1}{2}, f(x) + g(x) = \frac{1}{4} - \frac{1}{2} - 2$

$= -2\frac{3}{4}$

The range is $y \geq -2\frac{3}{4}$.

(b) $f(x) - g(x) = x - 5 - (x^2 + 3)$

$= -x^2 + x - 8$

Domain is the set of real numbers.

For the range you need to find the greatest value of the quadratic expression. Since there are no obvious factors of the quadratic expression, use the axis of symmetry of the parabola, $x = -\frac{b}{2a}$ to find where the greatest value occurs.

$x = -\frac{1}{(-2)} = \frac{1}{2}$

$x = \frac{1}{2}, f(x) - g(x) = -\frac{1}{4} + \frac{1}{2} - 8$

$= -7\frac{3}{4}$

The range is $y \leq -7\frac{3}{4}$.

(c) $f(x) + h(x) = x - 5 + \sqrt{x+4}$

No further simplification of this expression is possible. The domain of the new function will be the same as the domain of $h(x)$ as this is the more restrictive. Domain is $x \geq -4$.

The least value of $h(x)$ occurs when: $x = -4$

so the least value of $f(x) + h(x) = -4 - 5 + 0$

$= -9$

The range is $y \geq -9$.

(d) $k(x) - g(x) = x^3 - 2x^2 + 6 - (x^2 + 3)$
$= x^3 - 3x^2 + 3$

Domain is the set of real numbers.

Since $k(x) - g(x)$ is a polynomial of degree 3, the range is the set of real numbers.

(e) $g(x) - h(x) = x^2 + 3 - \sqrt{x+4}$

The domain of the new function will be the same as the domain of $h(x)$ as this is the more restrictive. Domain is $x \geq -4$.

The least value of $x^2 + 3$ occurs at $x = 0$ and is 3.

The least value of $\sqrt{x+4}$ occurs when $x = -4$ and is 0.

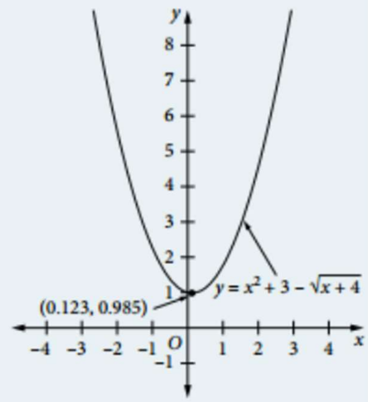
When $x = 0$, $g(x) - h(x) = 0 + 3 - 2 = 1$ and this appears to be the least value of $g(x) - h(x)$.

The only way to check this assertion at this stage is to draw a graph of the function.

The graph shows that the least value of $x^2 + 3 - \sqrt{x+4}$ occurs when $x = 0.123$.

The range of $x^2 + 3 - \sqrt{x+4}$ is then $y \geq 0.985$.

The initial answer was a good estimate.



Product and quotient of functions

The product of two functions, which may be written as $f(x) \times g(x)$ or $f(x) \cdot g(x)$, will have its domain determined by the more restrictive of the two original domains.

The quotient of two functions, $\frac{f(x)}{g(x)}$, will have its domain further restricted by the values of x for which $g(x) = 0$, because at those values $\frac{f(x)}{g(x)}$ is undefined.

Example 26

Given $f(x) = x - 3$ and $g(x) = x$, find expressions for each of the following functions, stating the domain and range in each case.

(a) $f(x) \cdot g(x)$ **(b)** $\frac{f(x)}{g(x)}$ **(c)** $\frac{g(x)}{f(x)}$

Solution

(a) $f(x) \cdot g(x) = (x - 3) \times x$
$= x^2 - 3x$

Domain: Real x as the quadratic expression exists for all values of x.

Since $f(x) \cdot g(x)$ is a quadratic polynomial, find the axis of symmetry:

$$x = -\frac{-3}{2} = \frac{3}{2}$$

$$x = \frac{3}{2}: f(x) \cdot g(x) = \frac{9}{4} - \frac{9}{2}$$

$$= -2\frac{1}{4}$$

This is the least value of the function as it is a concave up parabola.

Range is the set of real numbers $\geq -2\frac{1}{4}$.

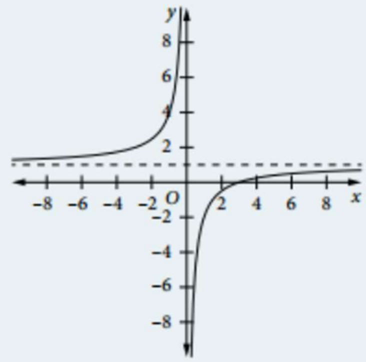
(b) $\frac{f(x)}{g(x)}=\frac{x-3}{x}=\frac{x}{x}-\frac{3}{x}=1-\frac{3}{x}$

Domain: Real x, $x \neq 0$.

As x gets larger, $x \to \infty$, $1-\frac{3}{x}$ gets smaller and approaches 1 from below.

As x gets smaller, $x \to -\infty$, $1-\frac{3}{x}$ gets smaller and approaches 1 from above.

Range is the set of real numbers except 1.



(c) $\frac{g(x)}{f(x)}=\frac{x}{x-3}$

$$=\frac{x-3+3}{x-3}$$

$$=1+\frac{3}{x-3}$$

The calculation here shows a way to simplify algebraic fractions when the power of the numerator is the same or greater than the denominator: create a factor of the denominator in the numerator, such as $(x-3)$ here, by adding and subtracting the same number (here $-3+3$). Because $-3+3$ cancels to zero, the expression has not really changed, but the term $(x-3)$ that appears can be divided to simplify.

Domain: Real x, $x \neq 3$.

For $x > 3$, as x gets larger, $x \to \infty$, $1+\frac{3}{x-3}$ gets smaller and approaches 1 from above.

For $x < 3$, as x gets smaller, $x \to -\infty$, $1+\frac{3}{x-3}$ gets larger and approaches 1 from below.

Range is the set of real numbers except 1.

Example 27

If $f(x) = x - 5$, $g(x) = x^2 + 3$, $h(x) = \sqrt{x+4}$ and $k(x) = x^3 - 2x^2 + 6$, find expressions for each of the following functions, stating the domain and range in each case.

(a) $f(x) \cdot g(x)$ **(b)** $g(x) \cdot h(x)$ **(c)** $\frac{f(x)}{g(x)}$ **(d)** $\frac{k(x)}{f(x)}$ **(e)** $\frac{h(x)}{f(x)}$

Solution

(a) $f(x) \cdot g(x) = (x-5)(x^2+3)$
$= x^3 - 5x^2 + 3x - 15$

Both $f(x)$ and $g(x)$ are defined for all real x.

The domain of $f(x) \cdot g(x)$ is all real x.

$f(x) \cdot g(x) = x^3 - 5x^2 + 3x - 15$ is a cubic polynomial.

The range is all real y.

(b) $g(x) \cdot h(x) = (x^2+3)\sqrt{x+4}$

$\sqrt{x+4}$ is only defined for $x \geq -4$.

The domain of $g(x) \cdot h(x)$ is real $x \geq -4$.

Since $x^2 + 3 \geq 3$ and $\sqrt{x+4} \geq 0$, the product $(x^2+3)\sqrt{x+4} \geq 0$.

The range is the set of non-negative real numbers, or $g(x) \cdot h(x) \geq 0$.

(c) $\frac{f(x)}{g(x)} = \frac{x-5}{x^2+3}$

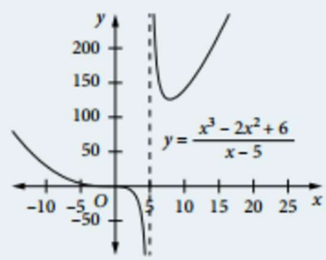
$x^2 + 3 \geq 3$ so the denominator is never zero. The domain of $\frac{f(x)}{g(x)}$ is real x.

The sign of $\frac{f(x)}{g(x)}$ is the same as the sign of $f(x)$ and since the denominator is never zero the range of the function is the set of real numbers.

(d) $\frac{k(x)}{f(x)} = \frac{x^3 - 2x^2 + 6}{x-5}$

$f(x) = 0$ when $x = 5$ so $\frac{k(x)}{f(x)}$ is undefined at $x = 5$. The numerator exists for all values of x so the domain is real x, $x \neq 5$.

For the range, it looks as though the answer is the set of real numbers as when $x > 5$, $\frac{k(x)}{f(x)} > 0$, and when $x < 5$ then $\frac{k(x)}{f(x)}$ can be positive or negative. Once again, drawing a graph of the function can confirm these answers.



(e) $\frac{h(x)}{f(x)} = \frac{\sqrt{x+4}}{x-5}$

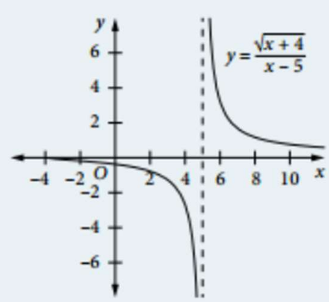
$f(x) = 0$ when $x = 5$ so $\frac{h(x)}{f(x)}$ is undefined at $x = 5$.

The numerator is defined for $x \geq -4$.

The domain of $\frac{h(x)}{f(x)}$ is $-4 \leq x < 5$ and $x > 5$.

The numerator is always positive or zero, the denominator can be positive or negative so the range of the function is the set of real numbers since $\frac{h(x)}{f(x)} = 0$ when $x = -4$.

The graph confirms this answer.



Using graphing software is a very effective way of finding domain and range when the functions are not straight polynomial functions.

Composite functions

$f(x) = (3x + 1)^2$ is a function in terms of x.

If $g(x) = 3x + 1$ then it is possible to write $f(x) = (g(x))^2$. A 'function of a function' like this is called a **composite function**.

It is sometimes written $f \circ g$ so if $f(x) = (g(x))^2$ and $g(x) = 3x + 1$ then $(f \circ g)(x) = (3x + 1)^2$.

The order of the functions is important, because $(g \circ f)(x) = 3(3x + 1)^2 + 1$ which is a very different function to $(f \circ g)(x)$.

In a composite function, the output of one function $g(x)$ has become the input of the other function: $f(g(x))$ or $(f \circ g)(x)$.

Example 28

If $f(x) = \sqrt{x+3}$, $g(x) = x^3 + 2$ and $h(x) = \frac{1}{x^2 - 1}$, find the expressions for:

(a) $f(g(x))$ **(b)** $f(h(x))$ **(c)** $g(h(x))$ **(d)** $g(f(x))$ **(e)** $h(f(x))$ **(f)** $h(g(x))$

Solution

(a) $f(g(x)) = \sqrt{\left(x^3+2\right)+3} = \sqrt{x^3+5}$

(b) $f(h(x)) = \sqrt{\frac{1}{x^2-1}+3} = \sqrt{\frac{1+3x^2-3}{x^2-1}} = \sqrt{\frac{3x^2-2}{x^2-1}}$

(c) $g(h(x)) = \left(\frac{1}{x^2-1}\right)^3 + 2 = \frac{1}{\left(x^2-1\right)^3} + 2$

(d) $g(f(x)) = \left(\sqrt{x+3}\right)^3 + 2 = (x+3)^{\frac{3}{2}} + 2$

(e) $h(f(x)) = \frac{1}{\left(\sqrt{x+3}\right)^2 - 1} = \frac{1}{x+3-1} = \frac{1}{x+2}$

(f) $h(g(x)) = \frac{1}{\left(x^3+2\right)^2 - 1} = \frac{1}{x^6 + 4x^3 + 3}$

With composite functions, you must be careful when finding the domain and range.

To find the domain of $f(x) = (3x + 1)^2$ you must first find the domain of $g(x) = 3x + 1$.

As $g(x)$ is defined for all x, then $f(x)$ is also defined for all x, because you can square any number. The range of $f(x)$ is $f(x) \geq 0$.

Example 29

Find the domain and range of $f(x)$ given the following.

(a) $f(x) = \sqrt{x-4}$ **(b)** $f(x) = \sqrt{x^3+8}$ **(c)** $f(x) = \frac{1}{x^2-1}$

Solution

(a) $f(x) = \sqrt{x-4}$: You can only take the square root of a non-negative number,
$x - 4 \geq 0$ gives $x \geq 4$.
Domain is $x \geq 4$
Range is $f(x) \geq 0$

(b) $f(x) = \sqrt{x^3+8}$: You can only take the square root of a non-negative number,
$x^3 + 8 \geq 0$ gives $x^3 \geq -8$ so $x \geq -2$ is the domain of $f(x)$.
Range is $f(x) \geq 0$

(c) $f(x) = \frac{1}{x-1}$: The denominator $x - 1 = 0$ at $x = 1$ so $f(x)$ is undefined for $x = 1$.
The domain of $f(x)$ is real x, $x \neq 1$.
Where $x > 1$ then $f(x) > 0$
Where $x < 1$ then $f(x) < 0$
$f(x)$ is never zero so the range of $f(x)$ is all real numbers, $f(x) \neq 0$.