ARITHMETIC SERIES

An **arithmetic series** is the sum of the terms in an arithmetic sequence. The series starts with the **first term**, which is usually denoted by a and is the first term of the sequence. The constant amount added each time to the term is called the **common difference** and is usually denoted by d.

Thus $S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d)$ is a series with a first term a and a common difference d.

- The nth term is $T_n = a + (n - 1)d$.
- The common difference is given by $d = T_2 - T_1 = T_3 - T_2 = \ldots = T_n - T_{n-1}$.
- Also note that $S_n = T_1 + T_2 + T_3 + \ldots + T_n$.

We call S_n the 'sum to n terms' of the series. This is a finite series.

Using sigma notation: $S_n = \sum_{k=1}^{n} (a + (k-1)d)$

Sum of an arithmetic sequence

You can obtain a formula for S_n so that you don't always have to add the terms to find the sum. First, note that the nth term of a series is sometimes called the 'last' term and denoted by l, so $T_1 = a$ and $T_n = l = a + (n - 1)d$.

Write: $S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 3)d) + (a + (n - 2)d) + (a + (n - 1)d)$ [1]

Let $l = a + (n - 1)d$, so: $S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$

Write this series in reverse: $S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a$ [2]

Add the two expressions [1] + [2]: $2S_n = (a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l)$

There are n factors of $(a + l)$: $2S_n = n(a + l)$

$$S_n = \frac{n}{2}(a + l)$$

But $l = a + (n - 1)d$, so: $S_n = \frac{n}{2}(a + a + (n-1)d)$

$$\therefore S_n = \frac{n}{2}(2a + (n-1)d)$$

It is also important to realise that $S_n = S_{n-1} + T_n$, so $T_n = S_n - S_{n-1}$, for $n > 1$.

Summary of formulae

- $a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d)$ is an arithmetic series
- d is the common difference
- $T_1 = a$
- $T_n = a + (n - 1)d$
- $S_n = \frac{n}{2}\{2a + (n-1)d\}$
- $S_n = \frac{n}{2}(a + l)$ where $l = a + (n - 1)d$
- $T_n = S_n - S_{n-1}$, $n > 1$

Example 15

Find the sum of the first twenty terms of the series $3 + 5 + 7 + \ldots$

Solution

$5 - 3 = 7 - 5 = 2$, so the series is arithmetic with $a = 3$, $d = 2$

$n = 20$: $S_n = \frac{n}{2}(2a + (n-1)d)$

$= \frac{20}{2}(6 + 19 \times 2)$

$= 440$

Example 16

How many terms of the series 4 + 7 + 10 + … must be taken to give a sum of 531?

Solution

$7 - 4 = 10 - 7 = 3$, so the series is arithmetic with $a = 4$, $d = 3$

$S_n = 531$: $\quad S_n = \frac{n}{2}(2a + (n-1)d)$

$$531 = \frac{n}{2}(8 + (n-1)3)$$

$$1062 = n(3n + 5)$$

$$3n^2 + 5n - 1062 = 0$$

$$(3n + 59)(n - 18) = 0$$

$$n = -19\tfrac{2}{3},\ 18$$

n must be a positive integer, so $n = 18$ and 18 terms must be taken.

Example 17

Find the sum of the first twenty terms of an arithmetic series, given that the tenth term is 39 and the sum of the first ten terms is 165.

Solution

$T_{10} = 39$: $\quad a + 9d = 39 \quad [1]$

$S_{10} = 165$: $\quad \frac{10}{2}(2a + 9d) = 165$

So: $\quad 2a + 9d = 33 \quad [2]$

[2] − [1]: $\quad a = -6$

Substitute into [1]: $\quad -6 + 9d = 39$

$$d = 5$$

$S_n = \frac{n}{2}(2a + (n-1)d)$: $\quad S_{20} = \frac{20}{2}(-12 + 19 \times 5) = 10(-12 + 95) = 830$

Example 18

Find the first three terms of the arithmetic series defined by $S_n = 2n^2 + n$.

Solution

$T_n = S_n - S_{n-1}$

$S_n = 2n^2 + n$

and $S_{n-1} = 2(n-1)^2 + (n-1)$

$S_{n-1} = 2n^2 - 3n + 1$

$\therefore\ T_n = 2n^2 + n - (2n^2 - 3n + 1)$

$T_n = 4n - 1$

$\therefore\ T_1 = 3,\ T_2 = 7,\ T_3 = 11$

Hence the series is $3 + 7 + 11 + \ldots$

Alternatively:

$S_n = 2n^2 + n$

$S_1 = 3 = T_1$

$S_2 = 10$

$T_2 = S_2 - S_1 = 10 - 3 = 7$

$S_3 = 21$

$T_3 = S_3 - S_2 = 21 - 10 = 11$

Hence the series is $3 + 7 + 11 + \ldots$