Example 38

Calculate the area of the region bounded by the curve $y = \frac{1}{x^2 + x}$, the x-axis and the ordinates $x = 2$ and $x = 3$.

Solution

Area $= \int_2^3 \frac{1}{x^2 + x}\,dx$

Use partial fractions: $\frac{1}{x^2 + x} = \frac{1}{x(x+1)} = \frac{a}{x} + \frac{b}{x+1}$

By inspection you can see that $a = 1$ and $b = -1$.

Or: $1 = a(x + 1) + bx$

$x = -1$: $1 = -b \Rightarrow b = -1$

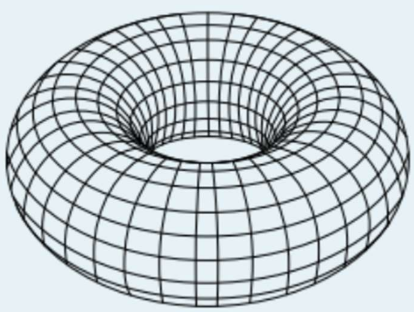
$x = 0$: $1 = a$

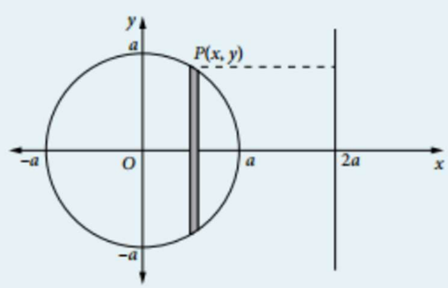
Hence

$$\begin{aligned}\text{Area} &= \int_2^3 \left(\frac{1}{x} - \frac{1}{x+1}\right)dx \\ &= \left[\log_e x - \log_e(x+1)\right]_2^3 \text{ since } x > 0 \\ &= \left[\log_e \frac{x}{x+1}\right]_2^3 \\ &= \log_e \frac{3}{4} - \log_e \frac{2}{3} \\ &= \log_e \frac{9}{8} \text{ units}^2\end{aligned}$$

Example 39

The region enclosed within the circle $x^2 + y^2 = a^2$ is rotated about the line $x = 2a$ to form a solid of revolution called a torus. (A torus is a ring-shaped object like a doughnut.)





The volume of this torus is given by $V = 8\pi a\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx - 4\pi\int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx$.

(a) Evaluate $8\pi a\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$.

(b) Evaluate $\int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx$.

(c) Hence find the volume of the torus.

Solution

(a) The integral $\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$ is the area of a semicircle of radius a.

Hence $8\pi a\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = 8\pi a\times\frac{\pi a^2}{2} = 4\pi^2a^3$.

(b) $$\int_{-a}^{a}x\sqrt{a^2-x^2}\,dx = -\frac{1}{2}\int_{-a}^{a}(-2x)\sqrt{a^2-x^2}\,dx$$
$$= -\frac{1}{3}\left[\left(a^2-x^2\right)^{\frac{3}{2}}\right]_{-a}^{a}$$
$$= 0$$

(c) $V = 4\pi^2a^3 - 4\pi\times 0 = 4\pi^2a^3$ units^3

In part **(a)**, if you did not recognise that the integral represented the area of a semicircle you would use the substitution $x = a\sin\theta$ to evaluate the integral. This would involve much more work, as shown below.

$x = a\sin\theta,\ dx = a\cos\theta.\quad x=-a,\ \theta=-\frac{\pi}{2}.\quad x=a,\ \theta=\frac{\pi}{2}.$

$$\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2\theta}\times a\cos\theta\,d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}a\cos\theta\times a\cos\theta\,d\theta$$
$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$
$$= \frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$
$$= \frac{a^2}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \frac{a^2}{2}\left(\frac{\pi}{2}+0-\left(-\frac{\pi}{2}\right)+0\right)$$
$$= \frac{\pi a^2}{2}$$

Hence $8\pi a\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = 8\pi a\times\frac{\pi a^2}{2} = 4\pi^2a^3$.

The solution of differential equations is all about integrating the given function and substituting the initial conditions to find a particular solution. You now have new integration skills, so this extends the types of differential equations that you can solve.

Example 40

Find the particular solution of $\frac{dy}{dx} = x^2\cos x$, given that $y = 0$ when $x = 0$.

Solution

$\frac{dy}{dx} = x^2\cos x:\quad y = \int x^2\cos x\,dx$

Use integration by parts: $y = x^2\sin x - \int 2x\sin x\,dx$

Use integration by parts: $y = x^2\sin x + 2x\cos x - \int 2\cos x\,dx$

$$= x^2\sin x + 2x\cos x - 2\sin x + C$$
$$x = 0:\ 0 = C$$

Hence the particular solution is $y = x^2\sin x + 2x\cos x - 2\sin x$.