PARAMETRIC AND CARTESIAN EQUATIONS

1 For the curves whose parametric equations are given, find:

(i) the Cartesian equation **(ii)** the vector equation.

(a) $x = 2t,\ y = t + 2,\ t \geq 0$

i) $x = 2t$ so $t = x/2$

$\therefore\ y = \frac{x}{2} + 2$ with $x \geqslant 0$

ii) The vector equation is:

$\vec{r}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath}$

$\vec{r}(t) = 2t\,\vec{\imath} + (t+2)\,\vec{\jmath}$

with $t \geqslant 0$

(b) $x = t,\ y = \frac{1}{t},\ t > 0$

i) $x = t$ so $y = \frac{1}{x}$ with $x > 0$

ii) The vector equation is

$\vec{r}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath}$

$\vec{r}(t) = t\,\vec{\imath} + \frac{1}{t}\,\vec{\jmath}$

with $t > 0$

(e) $x = u^3,\ y = 1 - u^2,\ -1 \leq u \leq 1$

i) $x = u^3$ so $u = \sqrt[3]{x} = x^{1/3}$

$\therefore\ y = 1 - u^2 = 1 - x^{2/3}$

with $-1 \leqslant x \leqslant 1$

ii) $\vec{r}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath}$

$\vec{r}(t) = u^3\,\vec{\imath} + (1-u^2)\,\vec{\jmath}$

with $-1 \leqslant u \leqslant 1$

(g) $x = \cos 2\theta,\ y = \cos\theta,\ 0 \leq \theta \leq 2\pi$

i) $y = \cos\theta$ so, as $\cos 2\theta = 2\cos^2\theta - 1$

then $x = 2y^2 - 1$

with $-1 \leqslant x \leqslant 1$

ii) $\vec{r}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath}$

$\vec{r}(t) = \cos 2\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath}$

with $0 \leqslant \theta \leqslant 2\pi$

PARAMETRIC AND CARTESIAN EQUATIONS

2 For the curves whose parametric equations are given, find:

(i) the Cartesian equation **(ii)** the vector equation.

(a) $x = \frac{2t}{1+t^2}, y = \frac{1-t^2}{1+t^2}, t \in R$

i) $x = \frac{2t}{(1+t^2)} = \sin A$ if $t = \tan\frac{A}{2}$

$y = \frac{1-t^2}{1+t^2} = \cos A$ if $t = \tan\frac{A}{2}$

$\therefore\ x^2+y^2 = \sin^2 A + \cos^2 A = 1$

$\therefore\ x^2+y^2 = 1$

Circle centre 0, radius 1

ii) $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

$\vec{r}(t) = \frac{2t}{1+t^2}\vec{i} + \frac{1-t^2}{1+t^2}\vec{j}$

with $t \in \mathbb{R}$

(b) $x = a\sin\phi, y = b\cos\phi, \phi \in R$

$x = a\sin\phi$ so $\frac{x}{a} = \sin\phi$

$y = b\cos\phi$ so $\frac{y}{b} = \cos\phi$

$\therefore\ \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

Ellipse centre 0, axes are x and y-axis, one diagonal is a, the other b.

ii) $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

$\vec{r}(t) = (a\sin\phi)\vec{i} + (b\cos\phi)\vec{j}$

with $\phi \in \mathbb{R}$

PARAMETRIC AND CARTESIAN EQUATIONS

3 For the curves whose vector equations are given, find:

(i) the parametric equation **(ii)** the Cartesian equation.

(e) $\underset{\sim}{r} = (2 - \sin\theta)\underset{\sim}{i} + (1 + \cos\theta)\underset{\sim}{j},\ 0 \le \theta \le 2\pi$

i) $x(t) = 2 - \sin\theta$

$y(t) = 1 + \cos\theta$

with $0 \leqslant \theta \leqslant 2\pi$

ii) $x(t) = 2 - \sin\theta$

so $x(t) - 2 = -\sin\theta$

$\sin\theta = 2 - x(t)$

and $\cos\theta = y(t) - 1$

$\sin^2\theta + \cos^2\theta = 1$

$\therefore (2-x)^2 + (y-1)^2 = 1$

or $(x-2)^2 + (y-1)^2 = 1$

Circle of centre $(2,1)$

radius 1

(f) $\underset{\sim}{r} = \left(t + \frac{1}{t}\right)\underset{\sim}{i} + \left(t - \frac{1}{t}\right)\underset{\sim}{j},\ t \ne 0$

i) $x(t) = t + \frac{1}{t}$

$y(t) = t - \frac{1}{t}$

with $t \neq 0$

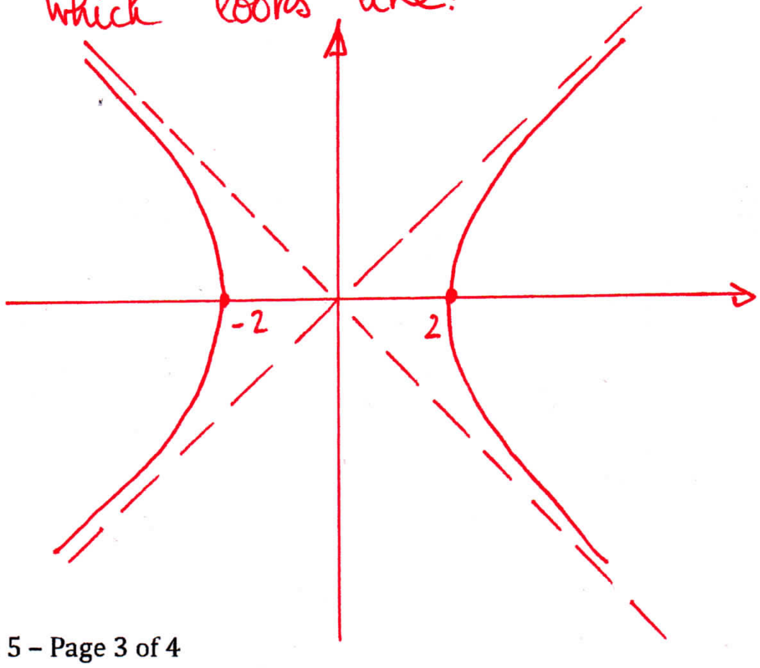
ii) $x^2(t) = \left(t + \frac{1}{t}\right)^2$

$x^2 = t^2 + 2 + \frac{1}{t^2}$

$y^2 = t^2 - 2 + \frac{1}{t^2}$

So $x^2 - y^2 = 4$

which looks like:



5 The position of a particle at any time, t, is $\underset{\sim}{r}(t) = (4\cos 3t)\underset{\sim}{i} + (4\sin 3t)\underset{\sim}{j}$.

(a) Show that the path is circular.

(b) Find the Cartesian equation of the path.

(c) Find the value of $|\underset{\sim}{r}|$.

a) $\vec{r}(t) = (4\cos 3t)\,\vec{i} + (4\sin 3t)\,\vec{j}$

so $x(t) = 4\cos 3t$ and $y(t) = 4\sin 3t$

$x^2 = 16\cos^2 3t$ $\quad y^2 = 16\sin^2 3t$

$\therefore\ x^2 + y^2 = 16 = 4^2$

Circle of centre $O(0,0)$ radius 4

b) as above.

c) $|\vec{r}| = 4$

[as $|\vec{r}| = \sqrt{(4\cos 3t)^2 + (4\sin 3t)^2}$

$|\vec{r}| = \sqrt{16\cos^2 3t + 16\sin^2 3t}$

$|\vec{r}| = \sqrt{16} = 4$]