MORE AREAS

2 Find the area of the region bounded by the parabola $y = x^2 + 2$, the x-axis and the lines $x = -1$ and $x = 2$.

First, we note that the curve is always above the x-axis, so the area is $\int_{-1}^{2} x^2+2\,dx$. indeed.

$$\int_{-1}^{2} x^2+2\,dx = \left[\frac{x^3}{3}+2x\right]_{-1}^{2} = \left(\frac{2^3}{3}+2\times2\right)-\left(\frac{(-1)^3}{3}+2(-1)\right)$$

$$= \frac{8}{3}+4-\left(\frac{-1}{3}-2\right) = \frac{8}{3}+4+\frac{1}{3}+2 = 6+3$$

$$= 9 \text{ square units}$$

4 Calculate the area of the region bounded by the curve $y = 4 - x^2$ and the x-axis.

the curve crosses the x-axis at $x=2$ and $x=-2$.

$$\int_{-2}^{2}(4-x^2)\,dx = 2\int_{0}^{2}(4-x^2)\,dx \quad \text{as the function is even.}$$

$$= 2\left[4x-\frac{x^3}{3}\right]_{0}^{2} = 2\left[4\times2-\frac{2^3}{3}\right]$$

$$= 2\left(8-\frac{8}{3}\right)$$

$$= \frac{32}{3} \text{ square units}$$

5 Which of these integrals will give the area of the region bounded by the curve $y = 16 - x^4$ and the x-axis? Indicate whether each answer is correct or incorrect.

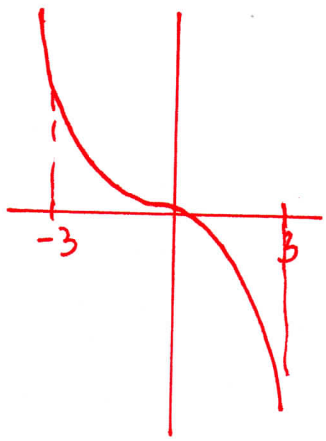
$\boxed{\textbf{(a)}}$ $\int_{-2}^{2}(16-x^4)\,dx$ **(b)** $\int_{-4}^{4}(16-x^4)\,dx$ $\boxed{\textbf{(c)}}$ $2\int_{0}^{2}(16-x^4)\,dx$ $\boxed{\textbf{(d)}}$ $\left|\int_{-2}^{2}(16-x^4)\,dx\right|$

6 Calculate the area of the region bounded by the curve $y = -x^3$, the x-axis and the ordinates $x = -3$ and $x = 3$.

$$\text{Area} = 2\int_{-3}^{0}(-x^3)\,dx = -2\int_{-3}^{0}x^3\,dx$$

$$\text{Area} = -2\left[\frac{x^4}{4}\right]_{-3}^{0} = -2\left[-\left(\frac{(-3)^4}{4}\right)\right]$$

$$\text{Area} = 2\times\frac{81}{4} = \frac{81}{2} = 40.5 \text{ units}^2$$



7 The value of the definite integral $\int_{-4}^{4} x^3\,dx$ is: **A** -128 $\boxed{\textbf{B}}$ 0 **C** 64 **D** 128

as $f(x) = x^3$ is odd.

8 Calculate the area of the region bounded by the graph of $f(x) = (x-2)^3$, the x-axis, $x = 2$ and $x = 3$.

Between $x = 2$ and $x = 3$, $f(x) \geqslant 0$.

So the area is $\int_{2}^{3}(x-2)^3\,dx = \left[\frac{(x-2)^4}{4}\right]_2^3$

$$= \frac{1^4}{4} - 0$$

$$= \frac{1}{4} \text{ units}^2$$

MORE AREAS

9 Find a positive number k such that the area of the region bounded by the graph of $f(x) = kx(2-x)^2$ and the x-axis is equal to 1 unit^2.

$f(x) = 0 \iff kx(2-x)^2 = 0 \Rightarrow x = 0$ or $x = 2$

For x between 0 and 2, $f(x) \geq 0$. So the area is the definite integral $\int_0^2 kx(2-x)^2\,dx$

$$\int_0^2 kx(2-x)^2\,dx = \int_0^2 kx(4-4x+x^2)\,dx = k\int_0^2 (4x-4x^2+x^3)\,dx$$

$$= k\left[4\frac{x^2}{2} - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_0^2 = k\left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4}\right]_0^2$$

$$= k\left[2\times 4 - \frac{4\times 8}{3} + \frac{16}{4}\right] = k\left[8 - \frac{32}{3} + 4\right]$$

$$= k \times \frac{4}{3}$$

so for the area to be equal to 1, we must have $k = 3/4$

10 For the graph of $f(x) = (x+1)(x-1)^2$, calculate:

(a) the area bounded by the curve, the x-axis, $x = 0$ and $x = 0.5$

(b) the area bounded by the curve and the x-axis

(c) the area to the right of the origin bounded by the curve and the coordinate axes.

a) Between $x = 0$ and $x = 0.5$, $f(x) \geq 0$. So the area is the definite integral $\int_0^{0.5} (x+1)(x-1)^2\,dx = \int_0^{0.5} (x+1)(x^2-2x+1)\,dx$.

$$A = \int_0^{0.5} (x^3 - 2x^2 + x + x^2 - 2x + 1)\,dx = \int_0^{0.5} (x^3 - x^2 - x + 1)\,dx$$

$$A = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} + x\right]_0^{0.5} = \frac{0.5^4}{4} - \frac{0.5^3}{3} - \frac{0.5^2}{2} + 0.5 = \frac{67}{192}$$

b) $$\int_{-1}^0 (x+1)(x-1)^2\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} + x\right]_{-1}^0 = -\left(\frac{1}{4} + \frac{1}{3} - \frac{1}{2} - 1\right) = \frac{11}{12}$$

$$\int_0^1 (x+1)(x-1)^2\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} + x\right]_0^1 = \frac{1}{4} - \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{12}$$

total $\frac{16}{12}$ or $4/3$

c) $$\int_0^1 (x+1)(x-1)^2\,dx = 5/12$$ as calculated above

12 Find the area of the region bounded by the curve $y = x(x-2)^2$ and the x-axis.

The curve cuts the x-axis at $x=0$ and $x=2$ and is positive between these values. So the area is the definite integral.

$$\int_0^2 x(x-2)^2\,dx = \int_0^2 (x^3 - 4x^2 + 4x)\,dx = \left[\frac{x^4}{4} - 4\frac{x^3}{3} + 4\frac{x^2}{2}\right]_0^2$$

$$= \frac{2^4}{4} - \frac{4\times 8}{3} + \frac{4\times 4}{2}$$

$$= 4 - \frac{32}{3} + 8 = \frac{4}{3} = 1\tfrac{1}{3} \text{ units}^2$$

13 Calculate the area of the region bounded by the curve $y = (x+1)(x-1)(x-3)$, the x-axis and the ordinates at $x = 0$ and $x = 2$.

roots -1 1 3

For $x=0$ $f(0) = 1 \times (-1) \times (-3) \geq 0$

whereas for $x=2$ $f(2) = 3 \times 1 \times (-1) < 0$

So the area is $\left[\int_0^1 f(x)\,dx - \int_1^2 f(x)\,dx\right]$

$$f(x) = (x^2-1)(x-3) = x^3 - 3x^2 - x + 3$$

$$\int_0^1 (x^3 - 3x^2 - x + 3)\,dx - \int_1^2 (x^3 - 3x^2 - x + 3)\,dx$$

$$= \left[\frac{x^4}{4} - 3\frac{x^3}{3} - \frac{x^2}{2} + 3x\right]_0^1 - \left[\frac{x^4}{4} - \frac{3x^3}{3} - \frac{x^2}{2} + 3x\right]_1^2$$

$$= \frac{1}{4} - 1 - \frac{1}{2} + 3 - \left[(4 - 8 - 2 + 6) - \left(\frac{1}{4} - 1 - \frac{1}{2} + 3\right)\right]$$

$$= 2\left(\frac{1}{4} - 1 - \frac{1}{2} + 3\right) - 0 = 3.5 \text{ units}^2$$