DOUBLE ANGLE FORMULAE

1 (a) By writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, write $\sin 3\theta$ in terms of $\sin\theta$.

(b) Hence write $\cos 3\theta$ in terms of $\cos\theta$. (c) Hence write $\tan 3\theta$ in terms of θ.

a) $\sin(3\theta) = \sin(2\theta+\theta) = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$\sin(3\theta) = 2\sin\theta\cos\theta \times \cos\theta + (1-2\sin^2\theta)\sin\theta$

$ = 2\sin\theta \times \cos^2\theta + \sin\theta - 2\sin^3\theta$

$ = 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$

$ = 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$\therefore \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

b) $\cos 3\theta = \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$\cos 3\theta = (2\cos^2\theta - 1)\times\cos\theta - (2\sin\theta\cos\theta)\times\sin\theta$

$\cos 3\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta$

$\cos 3\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta)$

$\cos 3\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

c) $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta} = \dfrac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta}$

$\tan 3\theta = \dfrac{\sin\theta}{\cos\theta} \times \left[\dfrac{3 - 4\sin^2\theta}{4\cos^2\theta - 3}\right] = \tan\theta \times \left[\dfrac{\dfrac{3}{\cos^2\theta} - \dfrac{4\sin^2\theta}{\cos^2\theta}}{\dfrac{4\cos^2\theta}{\cos^2\theta} - \dfrac{3}{\cos^2\theta}}\right]$

$\tan 3\theta = \tan\theta \times \left[\dfrac{3\sec^2\theta - 4\tan^2\theta}{4 - 3\sec^2\theta}\right] = \tan\theta \times \left[\dfrac{3(1+\tan^2\theta) - 4\tan^2\theta}{4 - 3(1+\tan^2\theta)}\right]$

$\tan 3\theta = \tan\theta \times \left[\dfrac{3 - \tan^2\theta}{1 - 3\tan^2\theta}\right] = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

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2 If $\sin\theta = \frac{3}{4}$, $90° < \theta < 180°$, evaluate (in surd form):

(a) $\sin 2\theta$ (b) $\cos 2\theta$ (c) $\tan 2\theta$. (d) In which quadrant is 2θ?

$90 < \theta < 180$ so $\cos\theta < 0$. (II quadrant)

As $\cos^2\theta + \sin^2\theta = 1$, then $\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}$

$\therefore \cos\theta = -\frac{\sqrt{7}}{4}$

a) $\sin 2\theta = 2\cos\theta\sin\theta = 2 \times \left(-\frac{\sqrt{7}}{4}\right) \times \frac{3}{4} = \frac{-3\sqrt{7}}{8}$

b) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 1 - 2\times\left(\frac{3}{4}\right)^2 = -\frac{1}{8}$

c) $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{-3\sqrt{7}}{8}}{\frac{-1}{8}} = 3\sqrt{7}$

d) $\sin 2\theta < 0$ and $\cos 2\theta < 0$

$\therefore$ III quadrant

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3 Simplify:

(a) $\dfrac{\sin 2A}{1+\cos 2A}$ (b) $\frac{1}{2}\sin 2\theta \tan\theta$ (c) $\cos^2 2\theta - \sin^2 2\theta$ (d) $\cos^2 30° - \sin^2 30°$

(e) $\sin 4x \cos 4x$ (f) $1 + \cos(180° + 2\theta)$ (g) $\sin x \cos x \cos 2x$ (h) $2\sin 2x \cos 2x$

a) $\dfrac{\sin 2A}{1+\cos 2A} = \dfrac{2\sin A\cos A}{1+[2\cos^2 A - 1]} = \dfrac{2\sin A\cos A}{2\cos^2 A} = \dfrac{\sin A}{\cos A} = \tan A$

b) $\dfrac{1}{2}\sin 2\theta \tan\theta = \dfrac{1}{2} \times 2\sin\theta\cos\theta \times \dfrac{\sin\theta}{\cos\theta} = \sin^2\theta$

c) $\cos^2 2\theta - \sin^2 2\theta = \cos 4\theta$

d) $\cos^2 30 - \sin^2 30 = \cos(2\times 30) = \cos 60 = 1/2$

e) $\sin 4x \cos 4x = \dfrac{1}{2} \times 2\sin 4x\cos 4x = \dfrac{1}{2}\sin 8x$.

f) $1 + \cos(180 + 2\theta) = 1 - \cos 2\theta$ as $\cos(\alpha + 180) = -\cos\alpha$

$= 1 - [1 - 2\sin^2\theta] = 2\sin^2\theta$

g) $\sin x \cos x \cos 2x = \dfrac{1}{2} \times 2\sin x\cos x \times \cos 2x$

$= \dfrac{1}{2} \times \sin 2x \cos 2x = \dfrac{1}{4} \times 2\sin 2x\cos 2x$

$= \dfrac{1}{4} \times \sin 4x$

h) $2\sin 2x \cos 2x = \sin 4x$

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3 Simplify:

(i) $(\sin\theta + \cos\theta)^2$ (j) $(\sin A - \cos A)^2$ (k) $\dfrac{2\tan\theta}{1-\tan^2\theta}$ for $\theta = 22.5°$ (l) $\sin^2 50° + \sin^2 40°$

(m) $\sin(45° - x)\cos(45° - x)$ (n) $\dfrac{1-\cos 2\theta}{1+\cos 2\theta}$ (o) $2\cos^2 3x - 1$

i) $(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + \sin 2\theta$

j) $(\sin A - \cos A)^2 = \sin^2 A + \cos^2 A - 2\sin A\cos A = 1 - \sin 2A$

k) $\dfrac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta = \tan(2\times 22.5) = \tan 45 = 1$

l) $\sin^2 50 + \sin^2 40 = \sin^2(90-40) + \sin^2 40$

$= \cos^2 40 + \sin^2 40 = 1$

m) $\sin(45-x)\cos(45-x) = \frac{1}{2}\times 2\sin(45-x)\cos(45-x)$

$= \frac{1}{2}\sin[2(45-x)] = \frac{1}{2}\sin(90-2x) = \frac{1}{2}\cos 2x$

n) $\dfrac{1-\cos 2\theta}{1+\cos 2\theta} = \dfrac{2\sin^2\theta}{2\cos^2\theta} = \tan^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

o) $2\cos^2 3x - 1 = \cos 6x$

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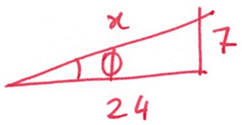
4 If $\sin\theta = \frac{3}{5}$, $\frac{\pi}{2} \le \theta \le \pi$ and $\tan\phi = \frac{7}{24}$, $0 \le \phi \le \frac{\pi}{2}$, find the value of:

(a) $\sin(\theta - \phi)$ (b) $\cos(\theta - \phi)$ (c) $\tan(\theta - \phi)$

1) $\sin\theta = \frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$ so II quadrant, $\therefore \cos\theta < 0$

$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$ so $\cos\theta = \frac{-4}{5}$

2) $\tan\phi = \frac{7}{24}$ and $0 < \phi \le \frac{\pi}{2}$ so I quadrant, $\sin\phi$ and $\cos\phi$ are both positive.



$x^2 = 7^2 + 24^2$ so $x^2 = 625$ $x = 25$

so $\sin\phi = \frac{7}{25}$ and $\cos\phi = \frac{24}{25}$

a) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

$= \frac{3}{5} \times \frac{24}{25} - \left(\frac{-4}{5}\right) \times \frac{7}{25} = \frac{72 + 28}{125} = \frac{4}{5}$

b) $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

$= \left(\frac{-4}{5}\right) \times \frac{24}{25} + \frac{7}{25} \times \frac{3}{5} = \frac{-96 + 21}{125} = -\frac{3}{5}$

c) $\tan(\theta - \phi) = \frac{\sin(\theta - \phi)}{\cos(\theta - \phi)} = \frac{4/5}{-3/5} = -\frac{4}{3}$

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5 Simplify:

(a) $1+\tan^2\left(\frac{\pi}{2}-\alpha\right)$ (b) $1-\cos^2(\pi+\theta)$ (c) $\sin\theta\cos\left(\frac{\pi}{2}-\theta\right)+\cos\theta\sin\left(\frac{\pi}{2}-\theta\right)$

(d) $2\cos^2\frac{\pi}{6}-1$ (e) $1-\sin\theta\cos\left(\frac{\pi}{2}-\theta\right)$ (f) $\sin(\pi-\theta)\cos\phi-\cos(\pi-\theta)\sin\phi$

a) $1+\tan^2\left(\frac{\pi}{2}-\alpha\right) = \sec^2\left(\frac{\pi}{2}-\alpha\right) = \frac{1}{\cos^2\left(\frac{\pi}{2}-\alpha\right)} = \frac{1}{\sin^2\alpha} = \operatorname{cosec}^2\alpha$

b) $1-\cos^2(\pi+\theta) = \sin^2(\pi+\theta) = \left[\sin(\pi+\theta)\right]^2 = \left[-\sin\theta\right]^2 = \sin^2\theta$

c) $\sin\theta\cos\left(\frac{\pi}{2}-\theta\right)+\cos\theta\sin\left(\frac{\pi}{2}-\theta\right) = \sin\theta\sin\theta+\cos\theta\cos\theta$

$= \sin^2\theta+\cos^2\theta = 1$

d) $2\cos^2\left(\frac{\pi}{6}\right)-1 = \cos\left(2\times\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

e) $1-\sin\theta\cos\left(\frac{\pi}{2}-\theta\right) = 1-\sin\theta\sin\theta = 1-\sin^2\theta = \cos^2\theta$

f) $\sin(\pi-\theta)\cos\phi-\cos(\pi-\theta)\sin\phi = \sin\left[(\pi-\theta)-\phi\right]$

$= \sin\left[\pi-(\theta+\phi)\right]$

$= \sin\left[\theta+\phi\right]$