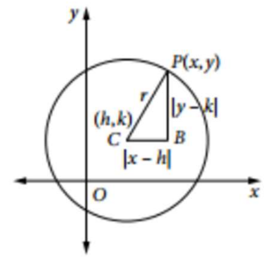
CIRCLES

A **circle** can be defined as the set of all points P in a plane at a given distance from a fixed point in the plane. The fixed point is the centre of the circle and the given distance is the **radius**.



Consider the circle of radius r units with its centre at $C(h, k)$. If P is a point with coordinates (x, y) on the circumference of this circle, then the distance of P from C is r units.

Applying Pythagoras' theorem to triangle CBP in the diagram gives:

$$BC^2 + BP^2 = CP^2$$

$$(x-h)^2 + (y-k)^2 = r^2$$

Thus the equation of the circle is given by $(x-h)^2 + (y-k)^2 = r^2$, with the values for x and y restricted:

- The set of values for x is given by $h - r \le x \le h + r$.
- The set of values for y is given by $k - r \le y \le k + r$.

If the centre of the circle is at the origin, then $h = 0$, $k = 0$ and the equation of the circle is $x^2 + y^2 = r^2$.

Example 14

Find the equation of the circle with centre $(-3, 4)$ and radius 6 units.

Solution

Use the result: $(x-h)^2 + (y-k)^2 = r^2$

Substitute $(-3, 4)$, $r = 6$: $(x+3)^2 + (y-4)^2 = 36$ is the equation of the circle.

Example 15

Find the coordinates of the centre and the length of the radius for the circle whose equation is $x^2 + y^2 - 4x + 10y + 14 = 0$.

Solution

Rewrite equation: $x^2 - 4x + y^2 + 10y = -14$

Complete the square for x and y: $x^2 - 4x + 4 + y^2 + 10y + 25 = -14 + 4 + 25$

Factorise: $(x-2)^2 + (y+5)^2 = 15$

The circle has its centre at $(2, -5)$ and has a radius of $\sqrt{15}$ units.

Example 16

Find the equation of the circle with centre (3, 4) that passes through the point (−1, 1).

Solution

Use the result: $(x-h)^2 + (y-k)^2 = r^2$

Centre is $(3, 4)$: $(x-3)^2 + (y-4)^2 = r^2$

$(-1, 1)$ satisfies equation: $(-4)^2 + (-3)^2 = r^2$

$r^2 = 25$

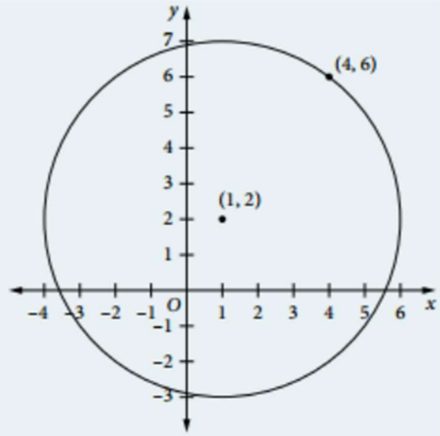
Equation of circle is $(x-3)^2 + (y-4)^2 = 25$.

Example 17

The diagram shows the graph of a circle with centre (1, 2) that passes through the point (4, 6). Find the equation of the circle:

(a) in the form $(x-h)^2+(y-k)^2=r^2$

(b) in general form.



Solution

(a) $C(1, 2)$, $P(4, 6)$

Radius of circle, $r = CP = \sqrt{(4-1)^2+(6-2)^2}$

$= \sqrt{3^2+4^2}$

$= 5$

Equation of the circle is: $(x-1)^2+(y-2)^2=25$

(b) In the general form the equation is: $x^2-2x+1+y^2-4y+4=25$

$x^2+y^2-2x-4y-20=0$

Semicircles

The equation $x^2 + y^2 = r^2$ can be written so that instead of being a relation, it becomes two functions that each represent a semicircle.

Rearrange the equation: $y^2 = r^2 - x^2$

Take square roots: $y = \pm\sqrt{r^2 - x^2}$

Thus the circle can be represented by two functions, $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$. Both these functions have the same domain, $x \in [-r, r]$.

- $y = \sqrt{r^2 - x^2}$ represents a semicircle in the upper half plane. The range is $y \in [0, r]$.
- $y = -\sqrt{r^2 - x^2}$ represents a semicircle in the lower half plane. The range is $y \in [-r, 0]$.

Example 18

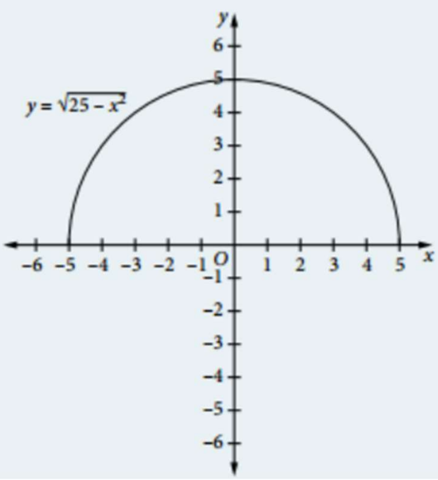
Given the equation $x^2 + y^2 = 25$, find the equations of the two functions that represent the semicircles that make up this circle. Sketch their graphs on separate diagrams.

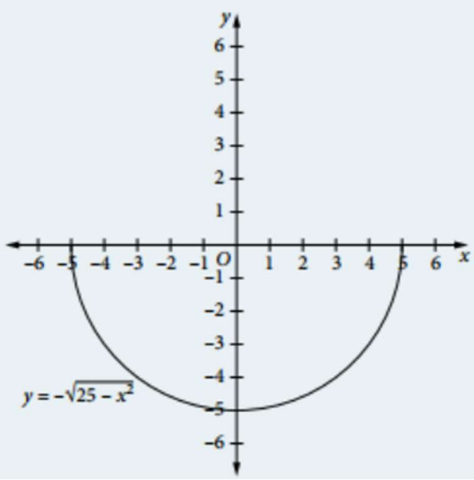
Solution

$x^2 + y^2 = 25$: $\quad y^2 = 25 - x^2$

$$y = \pm\sqrt{25 - x^2}$$

The equations of the semicircles are $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$.





The same approach for rewriting a circle relation as two functions is used when the centre of the circle is not the origin.

Given $(x - h)^2 + (y - k)^2 = r^2$:

Rearrange the equation: $(y - k)^2 = r^2 - (x - h)^2$

Take square roots: $y - k = \pm\sqrt{r^2 - (x - h)^2}$

$$y = k \pm \sqrt{r^2 - (x - h)^2}$$

The equations of the semicircles are $y = k + \sqrt{r^2 - (x - h)^2}$ and $y = k - \sqrt{r^2 - (x - h)^2}$.