PROBLEM SOLVING WITH DERIVATIVES

A function may not always be given algebraically. Sometimes you must interpret the information given to construct the function. Remember to show clearly what each variable represents. It often helps to draw a diagram of the situation being considered.

Example 13

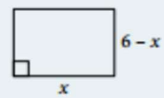
A piece of wire 12 cm long is bent in the shape of a rectangle. Find the maximum area of the rectangle.

Solution

Let $A(x)$ cm^2 be the area of the rectangle. Let the side lengths be x cm and y cm.

Because the wire is 12 cm long: $2x + 2y = 12 \quad \therefore y = (6 - x)$

Because the wire is 12 cm, the longest side of the rectangle cannot be longer than 6 cm.



Area of rectangle: $A(x) = x(6 - x)$ for $0 < x < 6$

$A(x) = 6x - x^2$

Differentiate: $A'(x) = 6 - 2x$

For stationary points, $A'(x) = 0$: $6 - 2x = 0$

$x = 3$

Differentiate again: $A''(x) = -2 < 0$ for all x

Hence $A(x)$ is concave down for all values of x in the domain. It has a maximum value when $x = 3$.

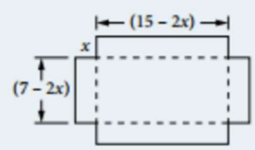
$A(3) = 3 \times 3 = 9\,\text{cm}^2$

The maximum area of the rectangle is 9 cm^2.

Example 14

A sheet of cardboard measures 15 cm by 7 cm. Four equal squares are cut out of the corners and the sides are turned up to form an open rectangular box. Find the edge length of the squares that were cut out to give the box a maximum volume.

Solution



Let the edge length of the squares that were cut out be x cm.

The dimensions of the base of the box will be $(15 - 2x)$ cm and $(7 - 2x)$ cm.

The height will be x cm.

Let the volume of the box be $V(x)$ cm^3. $\therefore V(x) = x(15 - 2x)(7 - 2x)$

$= 4x^3 - 44x^2 + 105x$

Differentiate: $V'(x) = 12x^2 - 88x + 105$

For stationary points, $V'(x) = 0$: $12x^2 - 88x + 105 = 0$

$(2x - 3)(6x - 35) = 0$

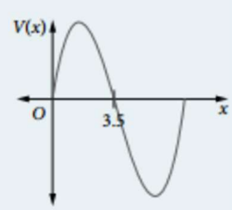
$\therefore x = 1\frac{1}{2}$ or $5\frac{5}{6}$

But x must be less than half the shortest side, i.e. $x < 3.5$, so we can disregard $x = 5\frac{5}{6}$. The only possible value for x is $x = 1.5$.

Use the first derivative test:

For $x < 1.5$, test $x = 1.4$: $V'(1.4) = 12 \times 1.4^2 - 88 \times 1.4 + 105 = 5.32 > 0$

For $x > 1.5$, test $x = 1.6$: $V'(1.6) = 12 \times 1.6^2 - 88 \times 1.6 + 105 = -5.08 < 0$



$V'(x)$ changes from positive to negative on passing through $x = 1.5$, so a maximum volume occurs when $x = 1.5$.

$V(1.5) = 1.5(15 - 3)(7 - 3) = 72\,\text{cm}^3$

A graph of $V(x)$ for $0 \le x \le 7$ helps to see what is happening.

Because $V(x) > 0$, the domain of the function is $0 < x < 3.5$.

The part of the graph below the x-axis is not relevant to the problem.