COMBINATIONS

Consider a set of four people, A, B, C and D.

In groups of two people at a time, they can be arranged as follows: AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC. That is $^4P_2 = 12$ arrangements.

However, the group AB is the same than the group BA, as the order in which they are combined does not matter; A and B can be arranged in 2! ways, but combined in only one way. Therefore the number of combinations (selections) of two people at a time is only 6.

The six different combinations are as follows: AB, AC, AD, BC, BD, CD.

This number, the number of combinations of 4 people taken 2 at a time, is represented by the symbol $^4C_2$ or $\binom{4}{2}$, and we can write $^4C_2 = \frac{^4P_2}{2!} = \frac{12}{2!} = 6$

Example 14

How many groups of three can be selected from the four people A, B, C and D?

Solution

The selections are: ABC, ABD, ACD, BCD.

Number of combinations $= {}^4C_3 = \frac{^4P_3}{3!} = \frac{4\times3\times2}{3\times2} = 4$

Number of combinations ^nC_r or $\binom{n}{r}$

The symbol ^nC_r or $\binom{n}{r}$ denotes the number of combinations of n different objects taken r at a time. Each combination consists of a group of r different elements that can be ordered in $r!$ ways.

$$^nC_r = \binom{n}{r} = \frac{^nP_r}{r!} = \frac{n!}{(n-r)!\,r!}$$

Note that ^nC_r and ^nP_r can be calculated with the functions nCr and nPr of the calculator.

Example 15

In how many ways can a group of four people be selected from ten people if:

(a) there are no restrictions

(b) the oldest person is included in the group

(c) the oldest person is excluded from the group?

(d) What proportion of all possible groups contain the oldest person?

Solution

(a) Number of ways $= {}^{10}C_4 = \frac{10!}{4!(10-4)!} = \frac{10\times9\times8\times7}{4\times3\times2\times1} = 210$

(b) Automatically including the oldest, the selection is now for three out of nine people.

Number of ways $= {}^9C_3 = \frac{9!}{3!(9-3)!} = \frac{9\times8\times7}{3\times2\times1} = 84$

(c) Excluding the oldest, the selection is now for four out of nine people.

Number of ways $= {}^9C_4 = \frac{9!}{4!(9-4)!} = \frac{9\times8\times7\times6}{4\times3\times2\times1} = 126$

(d) 84 out of the 210 possible sets contain the oldest person.

Proportion $= \frac{84}{210} = \frac{2}{5} = 40\%$

Important remark:

$$^{n}C_{n-r} = \frac{n!}{(n-(n-r))!\,(n-r)!} = \frac{n!}{(n-n+r)!\,(n-r)!} = \frac{n!}{r!\,(n-r)!} = {}^{n}C_{r}$$

Therefore $^{n}C_{r} = {}^{n}C_{n-r}$

That means that the number of combinations of n objects, taken r at a time, is equal to the number of combinations of n objects, taken $(n-r)$ at a time.

The reason for it is that for each set of r objects selected, there is a set containing $(n-r)$ objects left behind. Therefore, there must be the same number of sets containing $(n-r)$ objects as there are containing r objects.

Example 17

In how many ways can four runners and three swimmers be arranged in a row if there are eight runners and five swimmers to select from?

Solution

First select the total group of seven runners and swimmers, then arrange the group in a row:

- four runners and three swimmers can be selected in $^{8}C_{4} \times {}^{5}C_{3}$ ways.
- The group of seven can be arranged in 7! ways.

Number of arrangements $= {}^{8}C_{4} \times {}^{5}C_{3} \times 7!$

$$= \frac{8\times7\times6\times5}{4\times3\times2\times1} \times \frac{5\times4}{2\times1} \times 7! = 3\,528\,000$$

This answer is not $^{8}P_{4} \times {}^{5}P_{3}$, because the group has to be selected before it can be arranged. When selecting the group the order is not important, so the number of ways the group of seven can be selected is $^{8}C_{4} \times {}^{5}C_{3}$. After the group is selected they can then be arranged in a row.

Example 18

Prove that: $^{m+n}C_{r} = {}^{m}C_{r} + {}^{m}C_{r-1}{}^{n}C_{1} + {}^{m}C_{r-2}{}^{n}C_{2} + \ldots + {}^{m}C_{2}{}^{n}C_{r-2} + {}^{m}C_{1}{}^{n}C_{r-1} + {}^{n}C_{r}$

Solution

The symbol $^{m+n}C_{r}$ denotes the number of selections, each of r objects, when you select from two sets containing m and n objects respectively.

The r objects can be selected as follows (assuming r is not greater than either m or n):

r objects from m and no objects from n in $^{m}C_{r}$ ways

or $(r-1)$ objects from m and one object from n in $^{m}C_{r-1}{}^{n}C_{1}$ ways

or $(r-2)$ objects from m and two objects from n in $^{m}C_{r-2}{}^{n}C_{2}$ ways

and so on until:

no objects from m and r objects from n in $^{n}C_{r}$ ways.

These selections are mutually exclusive, so they are added to obtain the final result:

$$^{m+n}C_{r} = {}^{m}C_{r} + {}^{m}C_{r-1}{}^{n}C_{1} + {}^{m}C_{r-2}{}^{n}C_{2} + \ldots + {}^{m}C_{2}{}^{n}C_{r-2} + {}^{m}C_{1}{}^{n}C_{r-1} + {}^{n}C_{r}$$

Example 19

Prove that: ${}^{n+1}C_k = {}^nC_k + {}^nC_{k-1}$

Solution

Method 1

The symbol ${}^{n+1}C_k$ denotes the number of selections, each of k objects, when you select from two sets containing n objects and 1 object respectively.

The k objects can be selected as follows (assuming $k \le n$):

- k objects from the set of n and none from the set of one in ${}^nC_k{}^1C_0 = {}^nC_k$ ways
- $(k-1)$ objects from the set of n and the other one object in ${}^nC_{k-1}{}^1C_1 = {}^nC_{k-1}$ ways.

Note that ${}^1C_0 = 1$ and ${}^1C_1 = 1$.

These two selections are mutually exclusive, so they are added to obtain the final result:

$${}^{n+1}C_k = {}^nC_k + {}^nC_{k-1}$$

Method 2

Use the result for ${}^nC_r$ in factorial notation:

$$\begin{aligned}
{}^nC_k + {}^nC_{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\
&= n!\left(\frac{1}{k!(n-k)!} + \frac{1}{(k-1)!(n-k+1)!}\right) \\
&= n!\left(\frac{n-k+1}{k(k-1)!(n-k+1)!} + \frac{k}{k(k-1)!(n-k+1)!}\right) \quad \begin{cases} \text{from} & k! = k(k-1)! \\ \text{and} & (n-k+1)! = (n-k+1)\times(n-k)! \end{cases} \\
&= \frac{n!(n+1)}{k!(n+1-k)!} \qquad \textit{Note:} \quad (n-k+1) = (n+1-k) \\
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= {}^{n+1}C_k
\end{aligned}$$

Combinations of any number of objects

The symbol ${}^nC_r$ can also be described as representing the number of r-subsets in a given set, where $0 \le r \le n$.

If no specific value of r is stated, r can assume any value between 0 and n inclusive. Hence the number of subsets that each contain at least one element, i.e. the number of non-empty subsets in an n-set, is ${}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n$

From a set of n objects, each object can be dealt with in two ways: it can be included or it can be excluded. The number of ways of dealing with the n objects is thus $2 \times 2 \times 2 \times \ldots$ to n factors, or 2^n. But this includes the case where all n objects are excluded, so the number of combinations of at least one object is $2^n - 1$.

Therefore $\qquad {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n = 2^n - 1$

Example 20

How many selections can be made from five different books, taking any number of books at a time?

Solution

The books can be selected one at a time, two at a time, or up to five at a time.

$$\begin{aligned}
\text{Number of selections} &= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 \\
&= 5 + 10 + 10 + 5 + 1 \\
&= 31 = 2^5 - 1
\end{aligned}$$

Example 21

In how many ways can four people be divided into:

(a) an A pair and then a B pair **(b)** any pairs?

Solution

(a) Number of ways the A pair can be selected $= {}^4C_2 = 6$

Number of ways the B pair can now be selected $= 1$

$\therefore$ Number of ways an A pair and a B pair can be selected $= 6 \times 1 = 6$

After the first pair is selected, there is only one pair left for the B pair.

In this question you are distinguishing between the two pairs selected by calling them A and B.

(b) Number of ways of selecting two people from four $= {}^4C_2 = 6$

Let the people be W, X, Y, Z, so the six possible pairs are WX, WY, WZ, XY, XZ, YZ.

Unlike in part **(a)**, the order of selecting the two pairs now does not matter, so WX and YZ is not considered different to YZ and WX.

Thus the number of ways of dividing four people into two pairs $= \frac{{}^4C_2}{2!} = \frac{6}{2} = 3$

Example 22

In how many ways can six people be placed into three groups containing:

(a) three, two and one people respectively **(b)** two people in each group?

Solution

(a) Three people can be selected from the six people in ${}^6C_3$ ways.

From the remaining three people, two people can be selected in ${}^3C_2$ ways.

This leaves one person for the last group.

Number of ways $= {}^6C_3 \times {}^3C_2 \times 1$

$$= \frac{6!}{3!(6-3)!} \times \frac{3!}{2!(3-2)!} \times 1 = \frac{6 \times 5 \times 4}{2} = 60$$

(b) Two people can be selected from six people in ${}^6C_2$ ways.

From the remaining four people, two people can be selected in ${}^4C_2$ ways.

This leaves two people for the last group.

You might distinguish between the groups, e.g. by calling them X, Y, Z.

The number of ways people can be put into three distinct groups of $2 = {}^6C_2 \times {}^4C_2 \times 1 = 90$

If the groups are indistinguishable (i.e. do not have names), then you have to divide the total number of arrangements by 3! (the number of arrangements of the three groups).

Number of ways people can be put into any three groups of two $= \frac{{}^6C_2 \times {}^4C_2 \times 1}{3!} = \frac{90}{6} = 15$