THE GRADIENT AS A RATE OF CHANGE

Consider the linear function $y = f(x) = 3x + 2$. An increase of h in the value of x leads to an increase of $3h$ in the value of y. For any value of h, $h \neq 0$, y increases by an amount equal to three times the increase in x.

You can say that 'the rate of change of y with respect to x is 3'. The 3 is the gradient of the line. Because the gradient in this case is constant, y increases at a constant rate.

(This should remind you of differentiation from first principles, where the gradient of the function $= \frac{f(x+h) - f(x)}{h}$.)

Consider the function $y = f(x) = 2x^2$ at the point $(1, 2)$:

$$f(x) = 2x^2 \qquad f(1) = 2 \qquad f(1+h) = 2(1+h)^2 = 2 + 4h + 2h^2$$

$$k = f(1+h) - f(1) = 2(1+h)^2 - 2 = 4h + 2h^2$$

$$\frac{k}{h} = 4 + 2h,\ h \neq 0$$

Thus when x changes by an amount h, from 1 to $1 + h$, $f(x)$ changes by an amount $k = 4h + 2h^2$; that is, the value of the function changes by an amount that is $(4 + 2h)$ times the amount of the change in x.

The 'average rate of change of y with respect to x' $= 4 + 2h$. This rate varies with h.

As $h \to 0$, $4 + 2h \to 4$, i.e. $f'(1) = 4$. The number 4 in this case gives you:

(a) the gradient of the tangent at $x = 1$

(b) the rate of change of y with respect to x at $x = 1$.

This concept of the derivative as a 'rate of change' is very important in differential calculus. For example, there are many practical situations in which the change in a physical quantity depends on time:

- If a vessel is being filled with water, the volume V of the water in the vessel is a function of time. $\frac{dV}{dt}$ is the rate at which the volume changes, which may or may not be a constant rate.
- The population P of a town may increase or decrease with time. $\frac{dP}{dt}$ is the rate of change of population with respect to time.
- As a spherical balloon is being inflated, $\frac{dV}{dt}$ is the instantaneous rate of change of its volume with respect to time. Its spherical radius is also increasing, so $\frac{dr}{dt}$ is the rate of change of the radius with respect to time. Because a sphere's volume is given by $V = \frac{4}{3}\pi r^3$, we have $\frac{dV}{dr} = 4\pi r^2$, where $\frac{dV}{dr}$ measures the rate of change of volume with respect to the radius.

Note: When dealing with time t, you will only consider $t \geq 0$, so you will use the term **initially** to mean 'at time $t = 0$'.

Rates of change have many applications in physics and chemistry. Boyle's law for gases states that 'the pressure P of a given mass of gas varies inversely with the volume V', so that P is a function of V where:

$$P = \frac{k}{V} = kV^{-1}$$

$$\frac{dP}{dV} = -\frac{k}{V^2} = \text{rate of change of pressure with respect to volume.}$$

The negative sign indicates that an increase in V leads to a decrease in P.

Proportion

If two variables are in **direct proportion**, this means that their absolute values vary in the same direction: as the magnitude of the independent variable increases, the magnitude of the dependent variable will increase proportionally (and vice versa). Mathematically, the two variables are linear functions of each other (e.g. $y = kx$).

Inverse proportion means that the absolute values of the variables vary in opposite directions: as the magnitude of the independent variable increases, the magnitude of the dependent variable will decrease proportionally (and vice versa). Mathematically, the two variables are the inverse (reciprocal) of each other $\left(\text{e.g. } y = \frac{k}{x}\right)$.

Example 30

The volume V litres of water in a tank is given by $V = 4t + 30$, where t is in seconds.

(a) How much water is in the tank initially? **(b)** At what rate is water flowing into the tank?

Solution

(a) 'Initially' means $t = 0$, as we are dealing with time $t \geq 0$.

When $t = 0$: $V = 30$

Initially there is 30 L of water in the tank.

(b) If $V = 4t + 30$: $\frac{dV}{dt} = 4$

Water is flowing into the tank at a constant rate of 4 litres per second (L s^{-1}).