RESISTED MOTION

Whenever a body moves through a medium (such as air, water, oil etc.), it is subjected to a resistance that acts in the opposite direction to the motion. If the body falls vertically downwards, the resistance acts upwards; if the body is projected vertically upwards, the resistance acts downwards. (Of course the gravitational weight force mg always acts vertically downwards.)

In general, the faster the body moves, the greater the resistance. Air resistance is typically proportional to some power of the speed, so that air resistance $= kv^n$. This topic will consider this for $n = 1$ and $n = 2$, i.e. air resistance proportional to the speed of the body ($n = 1$) and air resistance proportional to the square of the speed of the body ($n = 2$). These are the two most useful models for air resistance.

It is important to realise that the effect of air resistance (or other resistance) is always one of retardation:

- **Resistance** is force acting in the opposite direction to the motion.
- **Retardation** is acceleration acting in the opposite direction to the motion.

In problems involving resisted motion, you should always define the positive direction to be the direction in which motion is actually occurring and take the origin O as the point from which the motion begins.

Resisted motion in a horizontal line

Example 21

A particle of mass m and initial speed v_0 moves on a horizontal surface against a resistance proportional to the square of the speed. Express the velocity in terms of the distance travelled.

Solution

Take the origin O as the particle's initial position. Take the particle's direction of motion as the positive direction.

Vertically, there is no motion (i.e. the particle moves always at the same horizontal level), so there is a zero resultant force acting in the vertical direction: $\therefore\ N - mg = 0$

Horizontally, there is motion. The only force acting horizontally is the resistance:

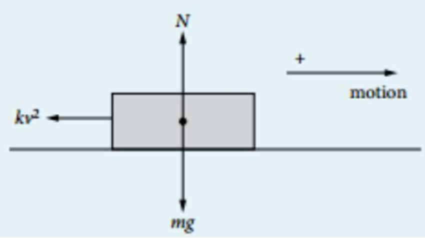
$$m\ddot{x} = -kv^2$$

$$\ddot{x} = -\frac{k}{m}v^2$$

$$v\frac{dv}{dx} = -\frac{k}{m}v^2$$

$$\frac{dv}{dx} = -\frac{k}{m}v$$

$$\therefore\ \frac{dx}{dv} = -\frac{m}{kv},\quad v \neq 0$$



$$\int_0^x dx = -\frac{m}{k}\int_{v_0}^{v}\frac{dv}{v} \qquad \text{(from start } x = 0,\ v = v_0 \text{ to end } x = x,\ v = v\text{)}$$

$$x = -\frac{m}{k}\left[\log_e v\right]_{v_0}^{v}$$

$$x = -\frac{m}{k}\log_e\frac{v}{v_0}$$

$$v = v_0 e^{-\frac{kx}{m}} \qquad \text{(making } v \text{ the subject)}$$

Motion of a particle falling downwards in a resisting medium with gravity

Example 22

A body of mass 5 kg is dropped from a great height under a constant gravitational acceleration $g\,\text{m s}^{-2}$ and air resistance proportional to the speed $v\,\text{m s}^{-1}$. If the constant of proportionality is $\frac{1}{8}$:

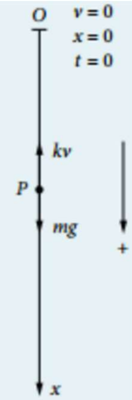
(a) find the velocity at time t

(b) sketch the velocity–time graph

(c) find the terminal (i.e. maximum) velocity

(d) find the distance the particle has fallen at time t.

Solution



(a) Take O as the point of release of the particle P. Take motion downwards as positive. There are two forces acting on P: its weight force of $5g$ acting downwards and its air resistance of $\frac{1}{8}v$ acting upwards (i.e. opposing the motion).

The resultant force on P is $5\ddot{x} = 5g - \frac{1}{8}v$ and this is the equation of motion:

$$\ddot{x} = g - \frac{v}{40}$$

$$\frac{dv}{dt} = \frac{40g - v}{40}$$

$$\frac{dt}{dv} = \frac{40}{40g - v}, \qquad v \neq 40g$$

$$\int_0^t dt = \int_0^v \frac{40}{40g - v}\,dv \qquad (\text{from } t = 0, v = 0 \text{ to } t = t, v = v)$$

$$t = -40\left[\log_e(40g - v)\right]_0^v, \quad 0 \leq v < 40g$$

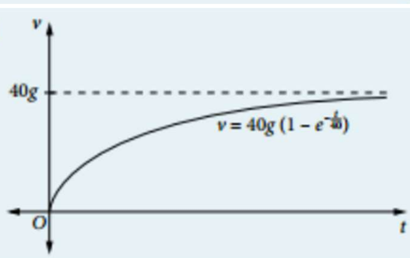
$$t = 40\log_e \frac{40g}{40g - v}$$

$$e^{\frac{t}{40}} = \frac{40g}{40g - v}$$

$$40g - v = 40ge^{-\frac{t}{40}}$$

$$v = 40g\left(1 - e^{-\frac{t}{40}}\right)$$

(b)



(c) As t increases, the gravitational force acting on P causes it to accelerate downwards. However, as the speed increases, the resistance also increases until eventually the downwards weight force and the resistance force are equal in magnitude but opposite in direction. At this stage the resultant force is zero: according to Newton's first law of motion, P will now move in a straight line at constant speed, i.e. the terminal velocity.

The terminal velocity occurs when the acceleration is equal to zero.

Method 1 (graphically):
In the velocity–time graph in part **(b)**, the horizontal asymptote at $v = 40g$ indicates the terminal velocity.

Method 2 (algebraically):

As $\ddot{x} = g - \frac{v}{40}$ the terminal velocity occurs when $g - \frac{v}{40} = 0$, i.e. when $v = 40g$.

(d) $v = 40g\left(1 - e^{-\frac{t}{40}}\right)$

$$\frac{dx}{dt} = 40g\left(1 - e^{-\frac{t}{40}}\right)$$

$$\int_0^x dx = 40g\int_0^t \left(1 - e^{-\frac{t}{40}}\right)dt \qquad \text{(from } x = 0, t = 0 \text{ to } x = x, t = t\text{)}$$

$$x = 40g\left[t + 40e^{-\frac{t}{40}}\right]_0^t$$

$$x = 40g\left(t + 40e^{-\frac{t}{40}}\right) - 40g \times 40$$

$$x = 40g\left(t + 40e^{-\frac{t}{40}} - 40\right)$$

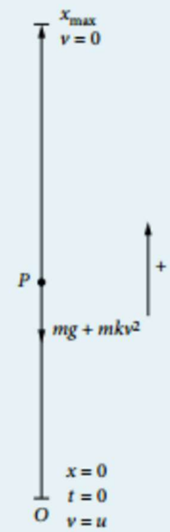
Motion of a particle moving upwards in a resisting medium with gravity

Example 23

A particle is projected vertically upwards with a velocity of u m s^{-1} under the influence of gravity and a resistance force proportional to the square of the speed. Find:

(a) the greatest height reached

(b) the time taken to reach the maximum height.



Solution

(a) Take O as the point of release of the particle P. Take motion upward as positive. There are two forces acting on P: its weight force of mg acting downwards and its resistance of mkv^2 acting downwards (i.e. opposing the motion).

(Note that although you do not have a numerical value for m, you write a factor of m into the constant of proportionality for the resistance. This will simplify later calculations where factors of m can be cancelled in the equation of motion.)

The resultant force on P is $m\ddot{x} = -mg - mkv^2$ and this is the equation of motion. You need to find x when $v = 0$:

$$\ddot{x} = -(g + kv^2)$$

$$v\frac{dv}{dx} = -(g + kv^2)$$

$$\frac{dv}{dx} = -\frac{g + kv^2}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g + kv^2}$$

$$\int_0^x dx = -\int_u^0 \frac{v}{g + kv^2}\,dv \text{ (from } x = 0, v = u \text{ to } x = x, v = 0)$$

$$x = -\frac{1}{2k}\left[\log_e(g + kv^2)\right]_u^0$$

$$x = -\frac{1}{2k}\log_e g + \frac{1}{2k}\log_e(g + ku^2)$$

$$x = \frac{1}{2k}\log_e\left(\frac{g + ku^2}{g}\right)$$

(b) You now need to find t when $v = 0$:

$$\ddot{x} = -(g + kv^2)$$

$$\frac{dv}{dt} = -(g + kv^2)$$

$$\int_0^t dt = -\int_u^0 \frac{1}{g + kv^2}\,dv \text{ (from } t = 0, v = u \text{ to } t = t, v = 0)$$

$$t = -\frac{1}{k}\int_u^0 \frac{1}{\left(\frac{g}{k} + v^2\right)}\,dv$$

$$t = -\frac{1}{k} \times \sqrt{\frac{k}{g}}\left[\tan^{-1}\left(\sqrt{\frac{k}{g}}\,v\right)\right]_u^0$$

$$t = -\frac{1}{\sqrt{kg}}\left(\tan^{-1} 0 - \tan^{-1}\sqrt{\frac{k}{g}}\,u\right)$$

$$t = \frac{1}{\sqrt{kg}}\tan^{-1}\sqrt{\frac{k}{g}}\,u$$ seconds, this is the time taken to reach the maximum height.

The next example looks at the motion of a particle that goes upwards and then downwards. This requires the solution to take a new O and a new positive direction when you analyse the downwards motion.

Example 24

A particle of mass 1 kg is projected vertically upwards from the ground at a speed of V m s^{-1}. The particle is acted on by both gravity and a resistance of magnitude $0.02v^2$, where v is the velocity of the particle at time t.

(a) Explain why the equation of motion while the particle is moving upwards is: $\ddot{x} = -(g + 0.02v^2)$

(b) Find the greatest height h reached by the particle.

(c) Find the time taken to reach this greatest height.

Having reached its maximum height, the particle falls back down towards its initial point of projection.

(d) Write the equation of motion for the downwards journey.

(e) Find the speed of the particle when it hits the ground.

(f) Determine whether the particle's speed on return is less than, equal to, or greater than its initial speed on projection.

Solution

(a) Take O as the point of projection and take upwards motion as positive.

The particle is acted on by gravitational force mg downwards and by resistance force $0.02v^2$ downwards (i.e. opposite to the motion).

Resultant force on the particle: $m\ddot{x} = -mg - 0.02v^2$

But $m = 1$, so: $\ddot{x} = -(g + 0.02v^2)$

(b)

$$\ddot{x} = -(g + 0.02v^2)$$

$$v\frac{dv}{dx} = -\left(\frac{50g + v^2}{50}\right)$$

$$\frac{dv}{dx} = -\left(\frac{50g + v^2}{50v}\right)$$

$$\frac{dx}{dv} = -\frac{50v}{50g + v^2}$$

$$\int_0^h dx = -\int_V^0 \frac{50v}{50g + v^2}\,dv \quad \text{(from } x = 0, v = V \text{ to } x = h, v = 0)$$

$$h = -25\left[\log_e(50g + v^2)\right]_V^0$$

$$h = -25\left(\log_e 50g - \log_e(50g + V^2)\right)$$

$$h = 25\log_e\left(\frac{50g + V^2}{50g}\right)$$

(c)

$$\frac{dv}{dt} = -\left(\frac{50g + v^2}{50}\right)$$

$$\frac{dt}{dv} = -\frac{50}{50g + v^2}$$

$$\int_0^t dt = -\int_V^0 \frac{50}{50g + v^2}\,dv \quad \text{(from } t = 0, v = V \text{ to } t = t, v = 0)$$

$$t = -\frac{50}{\sqrt{50g}}\left[\tan^{-1}\left(\frac{v}{\sqrt{50g}}\right)\right]_V^0$$

$$t = \sqrt{\frac{50}{g}}\tan^{-1}\left(\frac{V}{\sqrt{50g}}\right)$$

(d) For the downwards journey it is necessary to take a new O and a new positive direction. Remember:

> In problems involving resisted motion, you should always define the positive direction to be the direction in which motion is actually occurring and take the origin O as the point from which the motion begins.
>
> This simplifies the calculations required to solve the problem.

Take O as the maximum height and take downwards motion as positive.

The particle is acted on by gravitational force mg downwards and by resistance force $0.02v^2$ upwards (i.e. opposite to the motion).

Resultant force on the particle: $m\ddot{x} = mg - 0.02v^2$

But $m = 1$, so: $\ddot{x} = g - 0.02v^2$

(e) $v\frac{dv}{dx}=\frac{50g-v^2}{50}$

$$\frac{dv}{dx}=\frac{50g-v^2}{50v}$$

$$\frac{dx}{dv}=\frac{50v}{50g-v^2}$$

You need to find v when $x=h$:

$$\int_0^h dx=\int_0^v \frac{50v}{50g-v^2}\,dv$$

$$h=-25\Big[\log_e\left|50g-v^2\right|\Big]_0^v$$

$$h=25\log_e\left|\frac{50g}{50g-v^2}\right|$$

But from **(b)**: $h=25\log_e\left|\frac{50g+V^2}{50g}\right|$

$$\therefore\ 25\log_e\left|\frac{50g+V^2}{50g}\right|=25\log_e\left|\frac{50g}{50g-v^2}\right|$$

$$\frac{50g+V^2}{50g}=\frac{50g}{50g-v^2}$$

$$2500g^2+50gV^2-50gv^2-v^2V^2=2500g^2$$

$$50gV^2=v^2(50g+V^2)$$

$v^2=\frac{50gV^2}{50g+V^2}$ and so the speed on return is: $V\sqrt{\frac{50g}{50g+V^2}}$

(f) $V^2>0$, so $50g+V^2>50g$ and $\sqrt{\frac{50g}{50g+V^2}}<1$

$\therefore$ Speed on return is less than V, i.e. the speed on return is less than the speed of projection.