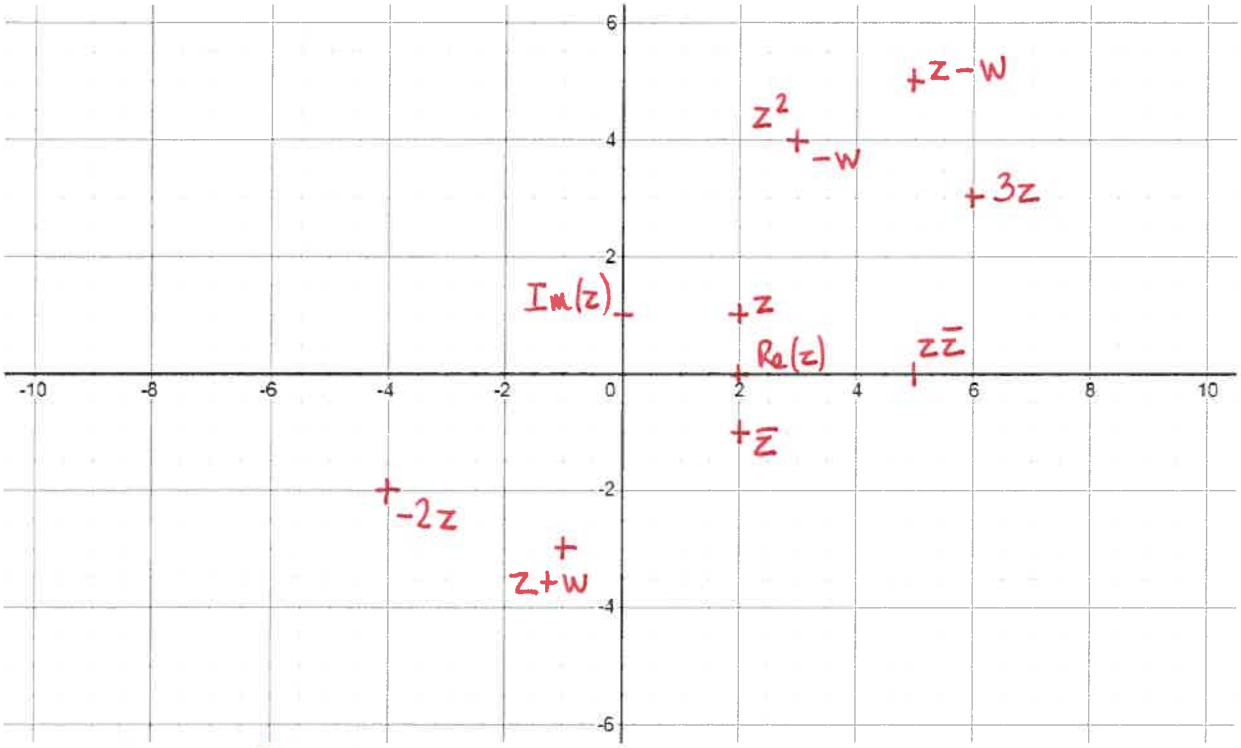
GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

1 If $z = 2 + i$ and $w = -3 - 4i$, represent each of the following on the complex plane.

(a)	z	**(b)**	$\bar{z}$	**(c)**	$z\bar{z}$	**(d)**	$3z$	**(e)**	$-2z$	**(f)**	$\frac{1}{z}$	**(g)**	$z+w$
(h)	$-w$	**(i)**	$z-w$	**(j)**	z^2	**(k)**	$\mathrm{Re}(z)$	**(l)**	$\mathrm{Im}(z)$				



c) $z\bar{z} = |z|^2 = 2^2 + 1^2 = 5$ (is a real number)

f) $\frac{1}{z} = \frac{1}{2+i} = \frac{2-i}{(2+i)(2-i)} = \frac{2-i}{4+1} = \frac{2}{5} - \frac{i}{5}$

k) $z^2 = (2+i)^2 = 4 - 1 + 4i = 3 + 4i$

2 If $z = 2\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$, then $z^4 =$ $2^4\left[\cos\left(4\times\left(-\frac{2\pi}{3}\right)\right) + i\sin\left(4\times\left(-\frac{2\pi}{3}\right)\right)\right]$

A $16\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$ **B** $16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ $= 16\left[\cos\left(-\frac{8\pi}{3}\right) + i\sin\left(-\frac{8\pi}{3}\right)\right]$

C $16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ **D** $16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$z^4 = 16\left[\cos\left(-\frac{2\pi}{3} - 2\pi\right) + i\sin\left(-\frac{2\pi}{3} - 2\pi\right)\right]$

$z^4 = 16\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$ so $\boxed{A}$

3 If $z = \bar{z}$, then $\arg z = \ldots$

A π **B** $\frac{\pi}{2}$ **C** 0 **D** 0 or π

$z = \bar{z}$ means $x + iy = x - iy$ so $iy = -iy$ so $2iy = 0$ so y must be 0. (i.e z is a real number) so it is on the x-axis. so either $\arg z = 0$ or $\arg z = \pi$ $\boxed{D}$

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4 Express each of the following in mod–arg form. (Give the argument in radians and in exact form.)

(a) $2-2i$ (b) $-\sqrt{3}+i$ (c) $-6-6i$ (d) $4i$ (e) -4

a) $|z| = \sqrt{2^2+2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

then, for the argument: $2\sqrt{2}\cos\theta = 2$ and $2\sqrt{2}\sin\theta = -2$

so $\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and $\sin\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

so $\theta = -\pi/4$ is the principal argument $z = 2\sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

b) $|z| = \sqrt{3+1} = \sqrt{4} = 2$

then $\begin{cases} 2\cos\theta = -\sqrt{3} \\ 2\sin\theta = 1 \end{cases}$ so $\begin{cases} \cos\theta = -\sqrt{3}/2 \\ \sin\theta = 1/2 \end{cases}$ so $\theta = \frac{5\pi}{6}$

$z = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(5\pi/6\right)\right]$

c) $|z| = \sqrt{6^2+6^2} = \sqrt{72} = 6\sqrt{2}$

then $\begin{cases} 6\sqrt{2}\cos\theta = -6 \\ 6\sqrt{2}\sin\theta = -6 \end{cases}$ or $\begin{cases} \cos\theta = -1/\sqrt{2} = -\frac{\sqrt{2}}{2} \\ \sin\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \end{cases}$ so $\theta = \frac{-3\pi}{4}$ is principal argument

d) $4i = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

e) $-4 = 4\left(\cos\pi + i\sin\pi\right)$

(f) $-3-\sqrt{3}i$ **(g)** $2\sqrt{3}-2i$ **(h)** $\sqrt{2}+\sqrt{2}i$

f) $|z| = \sqrt{3^2+3} = \sqrt{12} = 2\sqrt{3}$

then $\begin{cases} 2\sqrt{3}\cos\theta = -3 \\ 2\sqrt{3}\sin\theta = -\sqrt{3} \end{cases}$ or $\begin{cases} \cos\theta = -\frac{\sqrt{3}}{2} \\ \sin\theta = -\frac{1}{2} \end{cases}$ so $\theta = -\frac{5\pi}{6}$

$z = 2\sqrt{3}\left[\cos\left(-5\pi/6\right) + i\sin\left(-5\pi/6\right)\right]$

g) $|z| = \sqrt{4\times3+4} = \sqrt{16} = 4$

then $\begin{cases} 4\cos\theta = 2\sqrt{3} \\ 4\sin\theta = -2 \end{cases}$ or $\begin{cases} \cos\theta = \frac{\sqrt{3}}{2} \\ \sin\theta = -\frac{1}{2} \end{cases}$ so $\theta = -\pi/6$

$z = 4\left[\cos\left(-\pi/6\right) + i\sin\left(-\pi/6\right)\right]$

h) $|z| = \sqrt{2+2} = \sqrt{4} = 2$

then $\begin{cases} 2\cos\theta = \sqrt{2} \\ 2\sin\theta = \sqrt{2} \end{cases}$ or $\begin{cases} \cos\theta = \sqrt{2}/2 \\ \sin\theta = \sqrt{2}/2 \end{cases}$ $\theta = \pi/4$

$z = 2\left[\cos\left(\pi/4\right) + i\sin\left(\pi/4\right)\right]$

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6 For each of the following, find both zw and $\frac{z}{w}$ in mod–arg form.

(a) $z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $w = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ **(b)** $z = 5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, $w = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

a) $|zw| = |z| \times |w| = 4 \times 4 = 16$

$\arg(zw) = \arg(z) + \arg(w) = \pi/3 + \pi/6 = \frac{\pi}{2}$

so $zw = 16\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\pi/2\right)\right]$

$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{4}{4} = 1$

$\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w) = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

So $z/w = 1 \times \left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right] = \frac{\sqrt{3}}{2} + i \times \frac{1}{2}$

b) $|zw| = |z| \times |w| = 5 \times 3 = 15$

$\arg(zw) = \arg z + \arg w = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

$zw = 15\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{5}{3}$

$\arg\left(\frac{z}{w}\right) = \arg z - \arg w = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

So $\frac{z}{w} = \frac{5}{3}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$

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7 If $z = x + iy$, prove the following.

(a) $|z| = |\bar{z}|$ **(b)** $z\bar{z} = |z|^2$ **(c)** $z + \frac{|z|^2}{z} = 2\text{Re}(z)$

a) $|z| = \sqrt{x^2+y^2}$

whereas $|\bar{z}| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |z|$ so $|z| = |\bar{z}|$

b) $z\bar{z} = (x+iy)(x-iy) = x^2+y^2$

whereas $|z|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$ so $z\bar{z} = |z|^2$

c) $z + \frac{|z|^2}{z} = z + \frac{z\bar{z}}{z}$ (from b))

$= z + \bar{z}$

$= x+iy + x-iy = 2x = 2\ \text{Re}(z)$

11 If $z = r(\cos\theta + i\sin\theta)$, show that $\frac{z}{z^2+r^2}$ is real.

$$\frac{z}{z^2+r^2} = \frac{z}{[r(\cos\theta+i\sin\theta)]^2 + r^2}$$

$$= \frac{z}{r^2(\cos 2\theta + i\sin 2\theta) + r^2}$$ using De Moivre formula

$$= \frac{z}{r^2(2\cos^2\theta - 1 + i \times 2\sin\theta\cos\theta) + r^2}$$ using double angle formulas for $\cos 2\theta$ and $\sin 2\theta$

$$= \frac{z}{r^2(2\cos^2\theta) + i \times 2\sin\theta\cos\theta}$$

$$= \frac{z}{2r^2\cos\theta\,[\cos\theta + i\sin\theta]}$$

$$= \frac{r(\cos\theta+i\sin\theta)}{2r^2\cos\theta(\cos\theta+i\sin\theta)} = \frac{1}{2r\cos\theta}$$ which is real

14 Use de Moivre's theorem to prove that the conjugate of a power is equal to the power of the conjugate, i.e. let $z = r(\cos\theta + i\sin\theta)$ and prove that $\overline{z^n} = (\bar{z})^n$.

$\overline{z^n} = \overline{r^n[\cos(n\theta) + i\sin(n\theta)]}$

$\overline{z^n} = r^n[\cos(n\theta) - i\sin(n\theta)]$

whereas: $(\bar{z})^n = [r(\cos\theta - i\sin\theta)]^n$

$(\bar{z})^n = [r(\cos(-\theta) + i\sin(-\theta))]^n$ as $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

$(\bar{z})^n = [r^n(\cos(-n\theta) + i\sin(-n\theta)]$

$(\bar{z})^n = r^n[\cos(n\theta) - i\sin(n\theta)]$ $\therefore \overline{z^n} = (\bar{z})^n$

15 We have already proved (earlier and in question **14**) that:

- $z+\bar{z} = 2\,\mathrm{Re}(z)$ and $z-\bar{z} = 2\,\mathrm{Im}(z)\times i$
- the conjugate of a sum is equal to the sum of the conjugates $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$
- the conjugate of a difference is equal to the difference of the conjugates
- the conjugate of a product is equal to the product of the conjugates → $\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$
- the conjugate of a quotient is equal to the quotient of the conjugates
- the conjugate of a power is equal to the power of the conjugate.
- It is also obvious that the conjugate of a real number is itself, i.e. if $z = x + 0i$ then $\bar{z} = x - 0i = z$.

Use these properties of conjugates to answer the following.

(a) Show that $z^n + (\bar{z})^n = 2\,\mathrm{Re}(z^n)$.

(b) Simplify $(1+\sqrt{3}i)^{10} + (1-\sqrt{3}i)^{10}$.

a) $z^n + (\bar{z})^n = z^n + \overline{z^n}$ as we demonstrated that $(\bar{z})^n = \overline{z^n}$.

$= 2\,\mathrm{Re}(z^n)$ as $z + \bar{z} = 2\,\mathrm{Re}(z)$

b) $(1+\sqrt{3}i)^{10} + (1-\sqrt{3}i)^{10} = 2 \times \mathrm{Re}\left[(1+\sqrt{3}i)^{10}\right]$

if $z = 1+\sqrt{3}i$ then $|z| = \sqrt{1+3} = \sqrt{4} = 2$

$\begin{cases} 2\cos\theta = 1 \\ 2\sin\theta = \sqrt{3} \end{cases}$ or $\begin{cases} \cos\theta = 1/2 \\ \sin\theta = \sqrt{3}/2 \end{cases}$ so $\theta = \pi/3$

$1+\sqrt{3}i = 2\left[\cos\pi/3 + i\sin\pi/3\right]$

So $(1+\sqrt{3}i)^{10} + (1-\sqrt{3}i)^{10} = 2 \times \mathrm{Re}\left[\left(2\left(\cos\pi/3 + i\sin\pi/3\right)\right)^{10}\right]$

$= 2 \times \mathrm{Re}\left[2^{10}\left(\cos\left(\frac{10\pi}{3}\right) + i\sin\left(10\times\frac{\pi}{3}\right)\right)\right]$

$= 2\times 2^{10}\cos\frac{10\pi}{3}$

$= 2^{11}\cos\left(\frac{4\pi}{3}+\frac{6\pi}{3}\right) = 2^{11}\cos\left(\frac{4\pi}{3}\right) = -2^{11}\times\frac{1}{2}$

$= -2^{10} = -1024$

16 Consider the cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ for which all the coefficients a, b, c and d are real. Let the complex number z be a root of the equation $P(x) = 0$. Show that $\bar{z}$ is also a root of $P(x) = 0$.

We know that $P(z) = 0$

$P(\bar{z}) = a(\bar{z})^3 + b(\bar{z})^2 + c\bar{z} + d$

From Exercise 14, we know that $\overline{z^n} = (\bar{z})^n$

So $P(\bar{z}) = a\overline{z^3} + b\overline{z^2} + c\bar{z} + d$

Now $k\bar{z} = \overline{kz}$ where k is real

So $P(\bar{z}) = \overline{az^3} + \overline{bz^2} + \overline{cz} + d$

But $d = \bar{d}$ as d is real

So $P(\bar{z}) = \overline{az^3} + \overline{bz^2} + \overline{cz} + \bar{d}$

But $\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$

So $P(\bar{z}) = \overline{\underbrace{az^3 + bz^2 + cz + d}_{=0}}$

So $P(\bar{z}) = \bar{0} = 0$.