INTEGRATION OF $\sin^2 x$ AND $\cos^2 x$

1 Find: (a) $\int 2\cos^2 x\,dx$ (b) $\int 2\sin^2 x\,dx$ (c) $\int \sin^2\frac{x}{2}\,dx$

a) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$, $\therefore 2\cos^2 x = 1 + \cos 2x$

$\int 2\cos^2 x\,dx = \int (1+\cos 2x)\,dx = x + \frac{\sin 2x}{2} + C$

b) $2\sin^2 x = 1 - \cos 2x$,

$\therefore \int 2\sin^2 x\,dx = \int (1 - \cos 2x)\,dx = x - \frac{\sin 2x}{2} + C$

Alternative method, using the result from a)

$\int 2\cos^2 x\,dx + \int 2\sin^2 x\,dx = \int 2(\cos^2 x + \sin^2 x)\,dx = \int 2\,dx = 2x + C$

$\therefore \int 2\sin^2 x\,dx = 2x + C - \int 2\cos^2 x\,dx = 2x + C - x - \frac{\sin 2x}{2}$

$\therefore \int 2\sin^2 x\,dx = x - \frac{\sin 2x}{2} + C$

c) $\cos x = 1 - 2\sin^2\frac{x}{2}$ $\therefore \sin^2\frac{x}{2} = \frac{1-\cos x}{2}$

$\therefore \int \sin^2\frac{x}{2}\,dx = \int \frac{1-\cos x}{2}\,dx$

$= \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos x\,dx$

$= \frac{x}{2} - \frac{1}{2}\sin x + C$

INTEGRATION OF sin^2x AND cos^2x

1 Find: **(d)** $\int 2\cos^2\frac{x}{2}dx$ **(e)** $\int \sin^2 3x\,dx$ **(f)** $\int \cos^2 4x\,dx$

d) $\cos x = 2\cos^2\frac{x}{2} - 1 \quad \therefore \quad 2\cos^2\frac{x}{2} = 1 + \cos x$

$\int 2\cos^2\frac{x}{2}\,dx = \int 1 + \cos x\,dx = x + \sin x + C$

e) $\cos 6x = 1 - 2\sin^2 3x \quad \therefore \quad \sin^2 3x = \frac{1-\cos 6x}{2}$

$\int \sin^2 3x\,dx = \int \frac{1-\cos 6x}{2}\,dx = \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos 6x\,dx$

$= \frac{x}{2} - \frac{1}{2} \times \frac{\sin 6x}{6} + C = \frac{x}{2} - \frac{\sin 6x}{12} + C$

f) $\cos 8x = 2\cos^2 4x - 1 \quad \therefore \quad \cos^2 4x = \frac{1+\cos 8x}{2}$

$\int \cos^2 4x\,dx = \int \frac{1+\cos 8x}{2}\,dx$

$= \int \frac{1}{2}\,dx + \frac{1}{2}\int \cos 8x\,dx$

$= \frac{x}{2} + \frac{1}{2} \times \frac{\sin 8x}{8} + C$

$= \frac{x}{2} + \frac{\sin 8x}{16} + C$

INTEGRATION OF $\sin^2 x$ AND $\cos^2 x$

2 Evaluate: **(a)** $\int_0^{\frac{\pi}{2}} 2\sin^2 x\,dx$ **(b)** $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$ **(c)** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x\,dx$

a) Using result from 1) b)

$$\int_0^{\pi/2} 2\sin^2 x\,dx = \left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right) - \left(0 + \frac{\sin 2\times 0}{2}\right) = \frac{\pi}{2}$$

b) Using result from 1) b) $\int \sin^2 x\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$

$$\therefore \int_0^{\pi/4} \sin^2 x\,dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi/4}$$

$$= \left(\frac{\pi}{8} - \frac{\sin \pi/2}{4}\right) - \left(\frac{0}{2} - \frac{\sin 2\times 0}{2}\right) = \frac{\pi}{8} - \frac{1}{4}$$

c) First, we look for the primitive of $\cos^2 x$

$\cos 2x = 2\cos^2 x - 1 \qquad \therefore \cos^2 x = \frac{1 + \cos 2x}{2}$

$$\int \cos^2 x\,dx = \int \frac{1 + \cos 2x}{2}\,dx = \int \frac{1}{2}\,dx + \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{x}{2} + \frac{1}{2} \times \frac{\sin 2x}{2} + C = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

$$\therefore \int_{\pi/6}^{\pi/3} \cos^2 x\,dx = \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_{\pi/6}^{\pi/3}$$

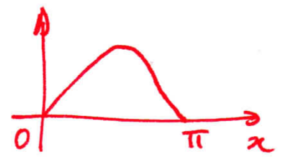
$$= \left(\frac{\pi}{6} + \frac{\sin(2\pi/3)}{4}\right) - \left(\frac{\pi}{12} + \frac{\sin(\pi/3)}{4}\right)$$

$$= \frac{\pi}{12} + \frac{1}{4}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \frac{\pi}{12}$$

INTEGRATION OF $\sin^2 x$ AND $\cos^2 x$

3 The region under the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated about the x-axis. The volume of the solid of revolution formed is given by:

A $\int_0^{\pi} \sin^2 x\,dx$ **B** $\pi\int_0^{\pi} \sin x\,dx$ $\boxed{\textbf{C}}$ $\pi\int_0^{\pi} \sin^2 x\,dx$ **D** $\pi\int_0^{\pi} \sin x^2\,dx$



Note: the curve is above the x-axis at all times so no problem with sign of integral.

4 The region under the curve $y = \cos x$ between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ is rotated about the x-axis. Find the volume of the solid of revolution formed.

Note: the curve is above the x-axis between $x = \pi/6$ and $x = \pi/3$, so there's no problem with the sign of the integral.

This volume is given by $V = \int_{\pi/6}^{\pi/3} \pi \cos^2 x\,dx$

$$V = \pi \int_{\pi/6}^{\pi/3} \cos^2 x\,dx$$

$\cos 2x = 2\cos^2 x - 1$

$\therefore \cos^2 x = \dfrac{1 + \cos 2x}{2}$

$$V = \frac{\pi}{2}\int_{\pi/6}^{\pi/3} (1 + \cos 2x)\,dx$$

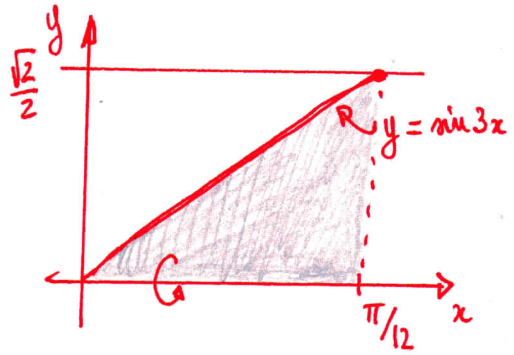
$$V = \frac{\pi}{2}\left[x + \frac{\sin 2x}{2}\right]_{\pi/6}^{\pi/3}$$

$$V = \frac{\pi}{2}\left[\left(\frac{\pi}{3} + \frac{1}{2} \times \sin\left(\frac{2\pi}{3}\right)\right) - \left(\frac{\pi}{6} + \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right)\right]$$

$$V = \frac{\pi}{2}\left[\frac{\pi}{6} + \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right]$$

$$V = \frac{\pi^2}{12} \text{ units}^3$$

9 The region bounded by the curves $y = \sin 3x$, the x-axis and the ordinate $x = \frac{\pi}{12}$ is rotated about the x-axis. Calculate the exact value of the volume of the solid of revolution formed.



when $x = \pi/12$ $\quad 3x = \frac{\pi}{4}$

this volume is $V = \int_0^{\pi/12} \pi \sin^2 3x \, dx$

$$\cos 6x = 1 - 2\sin^2 3x$$

$$\sin^2 3x = \frac{1 - \cos 6x}{2}$$

$$V = \frac{\pi}{2}\int_0^{\pi/12} (1 - \cos 6x)\, dx$$

$$V = \frac{\pi}{2}\left[x - \frac{\sin 6x}{6}\right]_0^{\pi/12}$$

$$V = \frac{\pi}{2}\left[\left(\frac{\pi}{12} - \frac{\sin(\pi/2)}{6}\right) - \left(0 - \frac{\sin 6 \times 0}{6}\right)\right]$$

$$V = \frac{\pi}{2}\left[\frac{\pi}{12} - \frac{1}{6}\right] = \frac{\pi^2}{24} - \frac{\pi}{12} \quad \text{units}^3$$

INTEGRATION OF sin^2x AND cos^2x

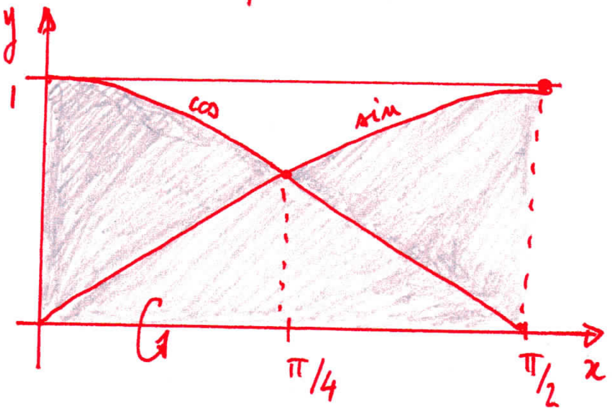
10 The region bounded by the curves $y = \sin x$, $y = \cos x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis.

(a) Find the point of intersection of the two curves.

(b) Calculate the exact value of the volume of the solid of revolution formed.

a) We look for the values of x between 0 and $\frac{\pi}{2}$ such that $\cos x = \sin x$. It's only $\pi/4$ as $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

b) When $x < \frac{\pi}{4}$, the curve $y = \cos x$ is above the curve $y = \sin x$, and vice-versa for $x > \frac{\pi}{4}$



$\therefore$ the volume is given by:

$$V = \int_0^{\pi/4} \pi \cos^2 x \, dx + \int_{\pi/4}^{\pi/2} \pi \sin^2 x \, dx$$

$$\cos 2x = 2\cos^2 x - 1$$
$$\text{——} = 1 - 2\sin^2 x$$

$$V = \pi \int_0^{\pi/4} \frac{1 + \cos 2x}{2} \, dx + \pi \int_{\pi/4}^{\pi/2} \frac{1 - \cos 2x}{2} \, dx$$

$$V = \pi \left[\frac{\pi}{8} + \frac{1}{2} \int_0^{\pi/4} \cos 2x \, dx \right] + \pi \left[\frac{\pi}{8} - \frac{1}{2} \int_{\pi/4}^{\pi/2} \cos 2x \, dx \right]$$

$$V = \frac{\pi^2}{4} + \frac{\pi}{2} \left[\left[\frac{\sin 2x}{2} \right]_0^{\pi/4} - \left[\frac{\sin 2x}{2} \right]_{\pi/4}^{\pi/2} \right]$$

$$V = \frac{\pi^2}{4} + \frac{\pi}{2} \left[\frac{1}{2} - \left(0 - \frac{1}{2} \right) \right]$$

$$V = \frac{\pi^2}{4} + \frac{\pi}{2} \times 1 = \frac{\pi}{4} \left[\pi + 2 \right]$$