GRAPHING RATIONAL ALGEBRAIC FUNCTIONS

Functions with the independent variable in the denominator generate curves that are not continuous and may have asymptotes. They may not have any turning points. You need to consider what happens to the function for very large positive and negative values of the variable.

Example 3

Sketch the graph of $y = \frac{1}{x-2}$.

Solution

Because $x - 2 \neq 0$, the function is not defined for $x = 2$, so at $x = 2$ there is a vertical asymptote.

For $x > 2$, $x - 2 > 0$, so $y > 0$. As $x \to 2$ from above, $x - 2$ is a very small positive number and so $y \to \infty$.
This can be written as: $x \to 2^+$, $y \to \infty$.

For $x < 2$, $x - 2 < 0$, so $y < 0$. As $x \to 2$ from below, $x - 2$ is a very small negative number and so $y \to -\infty$.
This can be written as: $x \to 2^-$, $y \to -\infty$.

The numerator of $y = \frac{1}{x-2}$ is never zero, so the curve does not cut the x-axis.

For $x = 0$, $y = -0.5$, so the y-intercept is -0.5.

As $x \to \infty$, $y \to 0$ from above; as $x \to -\infty$, $y \to 0$ from below. Thus $y = 0$ is a horizontal asymptote.

For stationary points, find $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{-1}{(x-2)^2}$, $x \neq 2$

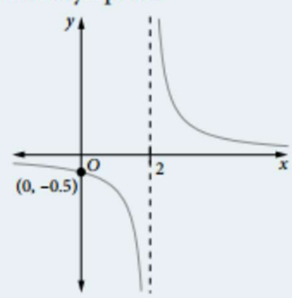
Hence $\frac{dy}{dx} < 0$ for all x in the domain, because $(x-2)^2 > 0$ in the domain.

Thus $y = \frac{1}{x-2}$ is a decreasing function in each part of its domain.

Also $\frac{dy}{dx} \neq 0$ in the domain, so there are no turning points.

For $x > 2$, $y > 0$; for $x < 2$, $y < 0$.

The curve is concave down for $x < 2$ and concave up for $x > 2$.



Example 4

Sketch the graph of $y = x + \frac{1}{x}$. For what values of x is the curve concave up? What is the range of the function?

Solution

$x \neq 0$: does not cut y-axis and $x = 0$ is a vertical asymptote.

$y = 0$: $x + \frac{1}{x} = 0$ or $\frac{x^2+1}{x} = 0$

Because $x^2 + 1 \neq 0$ for real x: does not cut x-axis.

As $x \to \infty$, $y \to x + [\text{very small amount}] \to x + 0$, so $y \to x$ from above.

As $x \to -\infty$, $y \to x - [\text{very small amount}] \to x - 0$, so $y \to x$ from below.

$\therefore$ $y = x$ is a sloping asymptote.

For stationary points, find $\frac{dy}{dx}$: $\frac{dy}{dx} = 1 - \frac{1}{x^2}, x \neq 0$

Hence for $\frac{dy}{dx} = 0$: $\frac{x^2-1}{x^2} = 0$, so $x = -1, 1$ (for which $y = -2, 2$)

$\therefore$ stationary points at $(-1,-2)$ and $(1,2)$.

Second derivative: $\frac{d^2y}{dx^2} = 0 - \frac{-2}{x^3} = \frac{2}{x^3}$

At $(-1,-2)$: $\frac{d^2y}{dx^2} = -2 < 0$

$\therefore$ $(-1,-2)$ is a local maximum turning point.

At $(1,2)$: $\frac{d^2y}{dx^2} = 2 > 0$

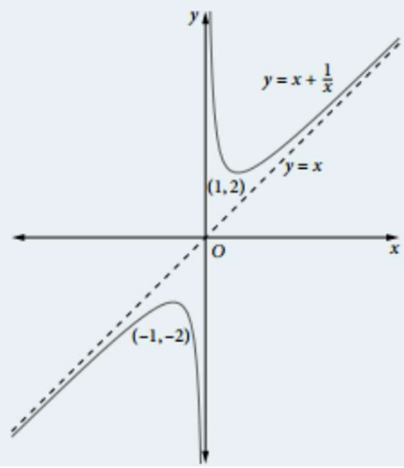
$\therefore$ $(1,2)$ is a local minimum turning point.

Because $\frac{d^2y}{dx^2} = \frac{2}{x^3}$ and $x \neq 0$, there are no points of inflection.

The curve is concave up for $x > 0$.

The range of the function is real $y, |y| \geq 2$.

Consider: why is the curve between the lines $x = 0$ and $y = x$?



Summary – rational algebraic function graphs

When sketching rational algebraic functions:

- identify any restrictions on the domain and the range
- find the intercepts on the coordinate axes where possible (it is usually easy to find the y-intercept, if it exists, but it is not always possible to find the x-intercept)
- use the symmetry properties of odd and even functions whenever possible
- find stationary points and determine their nature
- find asymptotes and use them to guide the shape of the curve. Asymptotes may be horizontal, vertical, sloping or occasionally another curve.

Remember that the shape of your graph sketches can be verified using graphing software.