SOLVING TRIGONOMETRIC EQUATIONS USING ANGLE FORMULAE AND THE t-FORMULAE

1 Solve for $0 \le x \le 2\pi$:

(a) $\sin\left(x+\frac{\pi}{3}\right)=\cos x$ **(b)** $\sin\left(x+\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}-x\right)$ (E) **(c)** $2\sin\left(x+\frac{\pi}{6}\right)=\sin x$

a) $\sin(a+b) = \sin a \cos b + \cos a \sin b$ so

$\sin\left(x+\frac{\pi}{3}\right) = \cos x \Leftrightarrow \sin x \cos\frac{\pi}{3} + \cos x \sin\frac{\pi}{3} = \cos x$

$\Leftrightarrow \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \cos x$

$\Leftrightarrow \frac{1}{2}\sin x = \cos x\left(1-\frac{\sqrt{3}}{2}\right) = \cos x\left(\frac{2-\sqrt{3}}{2}\right)$

$\Leftrightarrow \tan x = \left(2-\sqrt{3}\right)$ $x \simeq 0.262$ or $x \simeq 0.262+\pi$

$x \simeq 3.403$

b) (E) $\Leftrightarrow \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6} = \cos\left(\frac{\pi}{6}\right)\cos x + \sin\left(\frac{\pi}{6}\right)\sin x$

as $\cos(a+b) = \cos a \cos b - \sin a \sin b$

(E) $\Leftrightarrow \sin x\left[\frac{\sqrt{3}}{2}-\frac{1}{2}\right] + \cos x\left[\frac{1}{2}-\frac{\sqrt{3}}{2}\right] = 0$

(E) $\Leftrightarrow \sin x = \cos x \Leftrightarrow \tan x = 1$

so $x = \frac{\pi}{4}$ or $5\pi/4$

c) $2\left[\sin x \cos\left(\frac{\pi}{6}\right) + \cos x \sin\left(\frac{\pi}{6}\right)\right] = \sin x$

$\sin x\left[\sqrt{3}-1\right] + \cos x = 0$ so $\tan x = \frac{-1}{\sqrt{3}-1} = \frac{1}{1-\sqrt{3}}$

$x \simeq 2.203$ or $x \simeq \pi + 2.203$

$x \simeq 5.344$

2 Solve for $0 \le x \le 2\pi$:

(a) $2\cos x = \operatorname{cosec} x$ (E) **(b)** $4\sin x = \sec x$ (E) **(c)** $4\cos x = \sqrt{3}\operatorname{cosec} x$

a) $\operatorname{cosec} x = \frac{1}{\sin x}$ so (E) $\Leftrightarrow 2\cos x = \frac{1}{\sin x} \Leftrightarrow 2\cos x \sin x = 1$

$\Leftrightarrow \sin 2x = 1 = \sin\left(\frac{\pi}{2}\right)$

so $2x = \frac{\pi}{2} \times (-1)^n + n\pi$ is the general solution

$x = (-1)^n \frac{\pi}{4} + n\frac{\pi}{2}$

$n = 0$ gives $x = \frac{\pi}{4}$ $n = 1$ gives $x = \frac{\pi}{4}$ $n = 2$ gives $\frac{5\pi}{4}$

b) (E) $\Leftrightarrow 4\sin x = \frac{1}{\cos x} \Leftrightarrow 4\sin x \cos x = 1 \Leftrightarrow 2\sin 2x = 1$

or $\sin 2x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$ so the general solution is $2x = (-1)^n\frac{\pi}{6} + n\pi$

or $x = (-1)^n \frac{\pi}{12} + n\frac{\pi}{2}$ For $n = 0$ $x = \pi/12$

$n = 1$ gives $x = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{12}$

$n = 2$ gives $x = \pi + \pi/12 = \frac{13\pi}{12}$ $n = 3$ gives $x = \frac{3\pi}{2} - \frac{\pi}{12} = \frac{17\pi}{12}$

c) $4\cos x = \frac{\sqrt{3}}{\sin x} \Leftrightarrow 4\cos x \sin x = \sqrt{3} \Leftrightarrow 2\sin 2x = \sqrt{3}$

$\Leftrightarrow \sin 2x = \frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right)$

So the general solution is $2x = (-1)^n\frac{\pi}{3} + n\pi$

i.e. $x = (-1)^n \frac{\pi}{6} + n\frac{\pi}{2}$

For $n = 0$ gives $\boxed{x = \pi/6}$

$n = 1$ $x = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$ $\boxed{x = \frac{\pi}{3}}$ $n = 2$ gives $x = \pi + \frac{\pi}{6} = \boxed{\frac{7\pi}{6}}$

$n = 3$ gives $x = \frac{3\pi}{2} - \frac{\pi}{6} = \boxed{\frac{4\pi}{3}}$

(E)

8 Show that if $a^2 + b^2 < c^2$, the equation $a\cos\theta + b\sin\theta = c$ has no real roots.

We use the auxilliary angle method.

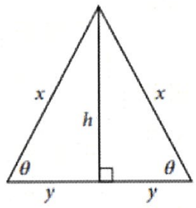
There exists an angle α such that $\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $\cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$

(E) $\Leftrightarrow$ $\sin\alpha\cos\theta + \cos\alpha\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$

(E) $\Leftrightarrow$ $\sin(\alpha+\theta) = \frac{c}{\sqrt{a^2+b^2}}$ This equation has a solution only if $\frac{c}{\sqrt{a^2+b^2}} \leqslant 1$

which is equivalent to $c^2 \leqslant a^2 + b^2$.

9 The equal sides of an isosceles triangle are x cm and the third side is $2y$ cm. The equal angles are each θ and the height of the triangle is h cm, as shown.
If the perimeter of the triangle is four times the height, find the size of the angles of the triangle to the nearest minute.



$2x + 2y = \text{Perimeter} = 4h$

$\Leftrightarrow$ $x + y = 2h$ Equation (1)

$\sin\theta = \frac{h}{x}$ and $\cos\theta = \frac{y}{x}$

Dividing both sides of (1) by x, we obtain $1 + \frac{y}{x} = 2\frac{h}{x}$

or $1 + \cos\theta = 2\sin\theta$

$\Leftrightarrow$ $2\sin\theta - \cos\theta = 1$ $\Leftrightarrow$ $\frac{2}{\sqrt5}\sin\theta - \frac{1}{\sqrt5}\cos\theta = \frac{1}{\sqrt5}$

So, with $\cos\alpha = \frac{2}{\sqrt5}$ and $\sin\alpha = \frac{1}{\sqrt5}$ (i.e $\alpha \simeq 26°33'54''$

it becomes $\cos\alpha\sin\theta - \sin\alpha\cos\theta = \frac{1}{\sqrt5} = \sin\alpha$

$\Leftrightarrow$ $\sin(\theta-\alpha) = \sin\alpha$ (as $\sin(A-B) = \sin A\cos B - \cos A\sin B$)

The general solution of this equation is $\theta - \alpha = \alpha(-1)^n + n \times 180$

$\Leftrightarrow$ $\theta = \alpha(-1)^n + \alpha + n \times 180$

$n = 0$ gives $\theta = 2\alpha = 2 \times 26°33'54'' \simeq 53°08'$

10 Solve each equation:

(a) $\tan^{-1}\left(\frac{x}{2}\right) - \tan^{-1}\left(\frac{x}{3}\right) = \tan^{-1}\left(\frac{1}{5}\right)$ **(b)** $\tan^{-1}(2x) + \tan^{-1}(3x) = \tan^{-1}(1)$

a) $\tan\left[\tan^{-1}\left(\frac{x}{2}\right) - \tan^{-1}\left(\frac{x}{3}\right)\right] = \tan\left(\tan^{-1}\left(\frac{1}{5}\right)\right)$ (E)

$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$ so (E) $\Leftrightarrow \frac{\frac{x}{2} - \frac{x}{3}}{1 + \frac{x}{2} \times \frac{x}{3}} = \frac{1}{5} \Leftrightarrow \frac{3x - 2x}{6 + x^2} = \frac{1}{5}$

$x^2 + 6 = 5x$ $x^2 - 5x + 6 = 0$ $\Delta = 25 - 4 \times 6 = 1$

$x_1 = \frac{5-1}{2} = 2$ or $x_2 = \frac{5+1}{2} = 3$

b) $\tan\left(\tan^{-1}(2x) + \tan^{-1}(3x)\right) = \tan\left(\tan^{-1}(1)\right)$

$\frac{2x + 3x}{1 - 2x \times 3x} = 1$ $5x = 1 - 6x^2$

$6x^2 + 5x - 1 = 0$

$\Delta = 25 + 4 \times 6 = 49 = 7^2$

$x_1 = \frac{-5+7}{2 \times 6} = \frac{2}{12} = \frac{1}{6}$

or $x_2 = \frac{-5-7}{2 \times 6} = \frac{-12}{12} = -1$

$t = \tan\frac{\theta}{2}$

13 Solve each equation using the t formulae, for $0° \leq \theta \leq 360°$.

(a) $2\sin\theta + \cos\theta = 1$ **(b)** $5\cos\theta + 3\sin\theta = 4$ **(c)** $2\,\mathrm{cosec}\,\theta - 4\cot\theta = 3$

a) $2 \times \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1 \Leftrightarrow 4t + \cancel{1} - t^2 = \cancel{1} + t^2$

$\Leftrightarrow 2t^2 - 4t = 0$

$\Leftrightarrow t(t-2) = 0$

So either $t = 0 = \tan\frac{\theta}{2}$ $\quad \frac{\theta}{2} = 0 + n\pi$

$\theta = 2n\pi$

$\theta = 0°$ or $\theta = 360°$

OR $t = 2 = \tan\frac{\theta}{2}$ $\quad \frac{\theta}{2} = \tan^{-1}2 + n \times 180$

$\theta = 2 \times \tan^{-1}2 + n \times 360$

$n = 0$ gives $\theta = 126°52'$

b) $5\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right) = 4$ or $5(1-t^2) + 6t = 4(1+t^2)$

$\Leftrightarrow -9t^2 + 6t + 1 = 0$

$\Delta = 36 - 4 \times (-9) = 72$ $\quad t = \frac{-6+\sqrt{72}}{2 \times (-9)}$ or $t = \frac{-6-\sqrt{72}}{(-18)}$

$\frac{\theta}{2} = -7.86 + 180n$ or $\frac{\theta}{2} = 38.82 + 180n$

$n = 1$ $\theta = 344°16'$ or for $n = 0$ $\theta = 77°38'$

c) $\frac{2}{\sin\theta} - \frac{4}{\tan\theta} = 3$ $\quad \frac{2(1+t^2)}{2t} - \frac{4(1-t^2)}{2t} = 3$

$\Leftrightarrow 2(1+t^2) - 4(1-t^2) = 6t \Leftrightarrow 1 + t^2 - 2(1-t^2) = 3t$

$\Leftrightarrow 3t^2 - 3t - 1 = 0$ $\quad \Delta = 9 - 4(-1) \times 3 = 21$

$t_1 = \frac{3+\sqrt{21}}{6} = \tan\frac{\theta}{2}$ so $\frac{\theta}{2} = \tan^{-1}\left(\frac{3+\sqrt{21}}{6}\right) + n \times 180$

$\theta = 2\tan^{-1}\left(\frac{3+\sqrt{21}}{6}\right) + n \times 360$

$n = 0$ gives $\theta = 103°18'$

$t_2 = \frac{3-\sqrt{21}}{6} = \tan\frac{\theta}{2}$ $\quad \theta = 2\tan^{-1}\left(\frac{3-\sqrt{21}}{6}\right) + n \times 360$

$n = 1$ gives $\theta = 330°26'$

16 Solve for $0 \le \theta \le \pi$.

(a) $\cos 3\theta = \sin\left(\frac{\pi}{4} - \theta\right)$ **(b)** $\sin 2\theta = \cos\left(\theta - \frac{\pi}{4}\right)$ **(c)** $\cos 2\theta = \sin\left(\theta + \frac{\pi}{4}\right)$

a) $\sin\left(3\theta + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{4} - \theta\right)$ so $3\theta + \frac{\pi}{2} = \left(\frac{\pi}{4} - \theta\right)(-1)^n + n\pi$

For $n=0$ $3\theta + \frac{\pi}{2} = \frac{\pi}{4} - \theta$ so $4\theta = -\frac{\pi}{4}$ $\theta = -\frac{\pi}{16}$ outside of $[0,\pi]$

$n=1$ $3\theta + \frac{\pi}{2} = \theta - \frac{\pi}{4} + \pi$ so $2\theta = -\frac{3\pi}{4} + \pi = \frac{\pi}{4}$ $\boxed{\theta = \frac{\pi}{8}}$

$n=2$ $3\theta + \frac{\pi}{2} = \left(\frac{\pi}{4} - \theta\right) + 2\pi$ so $4\theta = \frac{7\pi}{4}$ $\boxed{\theta = \frac{7\pi}{16}}$

$n=3$ $3\theta + \frac{\pi}{2} = \theta - \frac{\pi}{4} + 3\pi$ so $2\theta = \frac{9\pi}{4}$ $\cancel{\theta = \frac{9\pi}{8}}$ outside

$n=-1$ $3\theta + \frac{\pi}{2} = \theta - \frac{\pi}{4} - \pi$ so $2\theta =$ outside

$n=4$ $3\theta + \frac{\pi}{2} = \frac{\pi}{4} - \theta + 4\pi$ so $4\theta = \frac{15}{4}\pi$ so $\boxed{\theta = \frac{15\pi}{16}}$

b) $\sin 2\theta = \sin\left(\theta - \frac{\pi}{4} + \frac{\pi}{2}\right) = \sin\left(\theta + \frac{\pi}{4}\right)$ so $2\theta = (-1)^n\left(\theta + \frac{\pi}{4}\right) + n\pi$

$n=0$ gives $2\theta = \theta + \frac{\pi}{4}$ $\boxed{\theta = \frac{\pi}{4}}$

$n=1$ gives $2\theta = -\theta - \frac{\pi}{4} + \pi$ $3\theta = \frac{3\pi}{4}$ $\boxed{\theta = \frac{\pi}{4}}$ again

$n=2$ gives $2\theta = \left(\theta + \frac{\pi}{4}\right) + 2\pi$ $\theta = \frac{9\pi}{4}$ outside of $[0,\pi]$

$n=-1$ gives $2\theta = -\theta - \frac{\pi}{4} - \pi$ $\theta =$ outside of $[0,\pi]$

$n=3$ gives $3\theta = -\theta - \frac{\pi}{4} + 3\pi$ $3\theta = \frac{11}{4}\pi$ $\boxed{\theta = \frac{11}{12}\pi}$

c) $\cos 2\theta = \cos\left(\theta + \frac{3\pi}{4}\right)$ $2\theta = \pm\left(\theta + \frac{3\pi}{4}\right) + 2n\pi$

$n=0$ gives $2\theta = \pm\left(\theta + \frac{3\pi}{4}\right)$ $\boxed{\theta = \frac{3\pi}{4}}$ or $3\theta = -\frac{3\pi}{4}$ $\theta = -\frac{\pi}{4}$ outside of $[0,\pi]$

$n=1$ gives $2\theta = \pm\left(\theta + \frac{3\pi}{4}\right) + 2\pi$ $\theta = \frac{11\pi}{4}$ outside or $3\theta = \frac{5}{4}\pi$ $\boxed{\theta = \frac{5\pi}{12}}$

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$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

17 Solve for $0 \le x \le \pi$.

(a) $\sin 3x + \sin x = 0$ **(b)** $\sin 2x + \cos 3x = 0$ **(c)** $\tan 2x + \cot 3x = 0$

a) $2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)=0$ so either $\sin 2x = 0$ or $\cos x = 0$

$2\sin(2x)\cos x = 0$

$\cos x = 0$ when $\boxed{x=\pi/2}$.

and $\sin 2x = 0$ when $2x = n\pi$ i.e. $x = n\frac{\pi}{2}$ $\boxed{x=0}$ or $\boxed{x=\pi}$

b) $\sin 2x + \cos 3x = 0 \iff \sin(2x) + \sin\left(3x+\frac{\pi}{2}\right) = 0$

$\iff 2\sin\left(\frac{2x+3x+\pi/2}{2}\right)\cos\left(\frac{2x-3x-\pi/2}{2}\right)=0$

$\iff \sin\left(\frac{5x+\pi/2}{2}\right)\cos\left(\frac{-x-\pi/2}{2}\right)=0$

so either $\frac{5x+\frac{\pi}{2}}{2} = n\pi$ i.e. $5x+\frac{\pi}{2} = 2n\pi$ $x = -\pi/10$ $(n=0)$

$n=1$ $\boxed{x = 3\pi/10}$

$n=2$ $\boxed{x = \frac{7\pi}{10}}$ $n=3$ x outside of $[0,\pi]$

or $\frac{-x-\pi/2}{2} = \pm\frac{\pi}{2} + 2n\pi$ i.e. $-x = \pm\pi + 4n\pi + \frac{\pi}{2}$

$x = \pm\pi - 4n\pi - \frac{\pi}{2}$ $n=0$ $\boxed{x=\pi/2}$

others are outside $[0,\pi]$

c) $\tan 2x + \frac{1}{\tan 3x} = 0$ so $\tan 2x \tan 3x + 1 = 0$

$\iff \frac{\sin 2x}{\cos 2x}\frac{\sin 3x}{\cos 3x} + 1 = 0$

$\iff \sin 2x \sin 3x + \cos 2x \cos 3x = 0$

$\iff \cos(2x-3x) = 0$

$\iff \cos(-x) = 0$

$\iff \cos x = 0$ so $\boxed{x = \pi/2}$

indeed $\tan 2\times\frac{\pi}{2} = \tan\pi = 0$

and $\cot 3x = \frac{\cos 3x}{\sin 3x} = \frac{\cos(3\times\pi/2)}{\sin(3\times\pi/2)} = \frac{\cos(3\pi/2)}{\sin(3\pi/2)} = 0$