Applications to physical situations

Physics is full of problems in two- and three-dimensional space where vectors make the situation clearer. More advanced topics, such as electro-magnetic forces and waves, cannot be properly understood without vectors. At this stage, however, simple trigonometry will often solve problems quickly, and vectors may seem an unnecessary complication. The worked examples below use various methods, and each is done two ways. Readers should compare and contrast the methods used.

Without a great deal more physics, the only reasonable applications involve displacements, velocities, accelerations and forces. Some terminology is required, and some introduction to the relationship between force and acceleration.

Displacement and velocity

The first two worked examples involve only displacement and velocity.

Example 15 **8F**

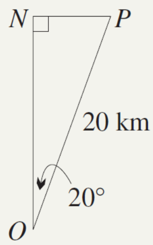
I walk 20 km in a direction N20°E. Find how far north I have gone:

a using a map of my journey,

b using projection vectors.

SOLUTION

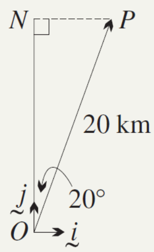
Let the walk begin at O and end at P.
Let N be the point north of O and west of P.



a By trigonometry, $ON = 20\cos 20° \doteqdot 18.8$.

b Using projections, let $\underset{\sim}{j}$ be a unit vector pointing north.

Then
$$|\overrightarrow{ON}| = \text{proj}_{\underset{\sim}{j}}\overrightarrow{OP} = (20\cos 20°)\underset{\sim}{j} \doteqdot 18.8\underset{\sim}{j}.$$



Hence I have gone 18.8 km north.

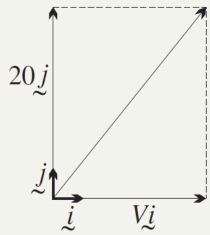
Example 16 — 8F

A ship leaves port and sails north-east in a straight line at an angle to the straight north-south coastline. Its speed along the coast is 20 km/h, and its speed in the water is 25 km/h (there is no current). Find its direction of motion and its speed away from the coast:

a using a velocity resolution diagram,

b using projection vectors.

SOLUTION



Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors east and north, and let V be the speed away from the coast.

Then the ship's velocity vector is $\underset{\sim}{v} = V\underset{\sim}{i} + 20\underset{\sim}{j}$.

We know that $V^2 + 20^2 = 25^2$,

so $V = 15$,

and the ship's velocity vector is $15\underset{\sim}{i} + 20\underset{\sim}{j}$.

Let θ be the acute angle between the ship's direction and shoreline.

a Using trigonometry, $\cos\theta = \dfrac{20}{25}$

$$\theta \doteqdot 37°.$$

b Using projections, velocity along the coast $= \text{proj}_{\underset{\sim}{j}}\underset{\sim}{v}$

$$20\underset{\sim}{j} = (|\underset{\sim}{v}|\cos\theta)\underset{\sim}{j}$$

$$20\underset{\sim}{j} = (25\cos\theta)\underset{\sim}{j}$$

$$\cos\theta = \tfrac{20}{25}$$

$$\theta \doteqdot 37°.$$

Hence the ship is travelling about N37°E, leaving the coast at 15 km/h.

APPLICATIONS TO PHYSICAL SITUATIONS (CAMBRIDGE)

The resultant of two vectors

The sum $\underset{\sim}{u} + \underset{\sim}{v}$ of two vectors is often called the *resultant* of the two vectors. This terminology is used particularly for the sum of two forces, when the sum or resultant of two forces can be regarded as a single force acting on the object, as in the next worked example.

Example 17 **8F**

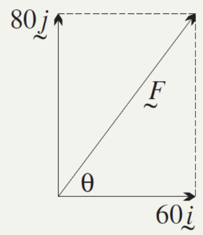
Peter and Paul are pulling a large box using ropes. They can never cooperate, and they end up pulling the box in different directions, Peter pulling east with a force of 60 newtons, and Paul pulling north with force of 80 newtons. Find the resultant force:

a using a forces diagram,

b using projection vectors.

SOLUTION

Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors east and north.



Then the resultant force is $\quad \underset{\sim}{F} = 60\underset{\sim}{i} + 80\underset{\sim}{j}$,

so $\quad |\underset{\sim}{F}| = 100$.

Let θ be the acute angle between $\underset{\sim}{i}$ and $\underset{\sim}{F}$.

a Using trigonometry,
$$\tan\theta = \tfrac{80}{60}$$
$$\theta \doteqdot 53°.$$

b Using projections,
$$\text{proj}_{\underset{\sim}{i}}\,\underset{\sim}{F} = 60\underset{\sim}{i}$$
$$(100\cos\theta)\,\underset{\sim}{i} = 60$$
$$\cos\theta = \tfrac{60}{100}$$
$$\theta \doteqdot 53°$$

Hence the resultant force is about 100 newtons in a direction N37°E.

Forces and their units

Newton's second law of motion states that

$F = ma$, $\quad$ meaning that $\quad$ force = mass × acceleration.

We will have a great deal more to say about acceleration in the next two chapters, but this law is needed now so that the units of force can be introduced — these units are called 'newtons', with symbol N, in honour of Sir Isaac Newton.

In words, Newton's second law says that if a body of mass m kg is accelerating at a m/s^2, then the sum of all the forces acting on the body has magnitude $F = ma$ newtons, and acts in the same direction as the acceleration. Thus 1 newton is the force required to accelerate a body at a rate of 1 m/s^2.

There is a second unit of force called 'kilograms weight'. A mass of m kilograms that is free to move is pulled downwards by gravity with a force that accelerates it at about 10 m/s^2. The physical value has symbol g, and a better approximation is $g = 9.8\ \text{m/s}^2$ ($g = 9.832\ \text{m/s}^2$ at the poles and $g = 9.780\ \text{m/s}^2$ at the equator).

This means that the downwards gravitational force on a mass m kg is mg newtons.

In particular, because $g \doteqdot 10\ \text{m/s}^2$, a force of 1 newton is about the downwards force that you feel when you hold a 100 g apple in your open hand.

19 NEWTON'S SECOND LAW AND THE UNITS OF FORCE

- One newton, written in symbols as 1 N, is the force required to accelerate a body at a rate of $1\ \text{m/s}^2$.
- Newton's second law of motion says that
  $$F = ma.$$
  'If a body of mass m kg is accelerating at $a\ \text{m/s}^2$, then the sum of all the forces acting on the body has magnitude $F = ma$ newtons, and acts in the same direction as the acceleration.'
- One kilogram weight is the downward force due to gravity on a mass of 1 kg at the Earth's surface.
- Acceleration due to gravity at the Earth's surface has the symbol g, whose approximate value is $9.8\ \text{m/s}^2$ (or $10\ \text{m/s}^2$ in round figures).
- One newton is therefore about $\frac{1}{10}$ kg weight — about the downward force due to gravity of a 100-gram apple on your open hand.

Example 18 — 8F

A parcel P of mass 8 kg is resting on a plane inclined at 30° to the horizontal.

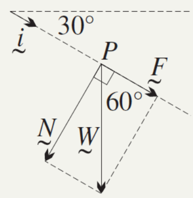
a What component of the gravitation force is acting on the parcel in the direction down the plane? Answer the question:

i using a diagram of forces, **ii** using projection vectors.

b What is the force in newtons, taking $g \doteqdot 9.8\ \text{m/s}^2$?

c What frictional force is acting on the parcel in the direction up the plane?

SOLUTION



Let $\underset{\sim}{W}$ be the weight of the parcel, represented as a vector down from P.

a i Regard the weight as the resultant of a force $\underset{\sim}{F}$ down the plane, and a force $\underset{\sim}{N}$ normal to the plane.

By simple trigonometry,
$$|\underset{\sim}{F}| = |\underset{\sim}{W}| \cos 60° = 8 \times \tfrac{1}{2} = 4.$$

ii Let $\underset{\sim}{W}$ be the weight of the parcel, represented as a vector down from P.

Let $\underset{\sim}{i}$ be a unit vector down the plane.

Then the force down the plane is the projection of $\underset{\sim}{W}$ onto i,
$$\underset{\sim}{F} = \text{proj}_{\underset{\sim}{i}}\underset{\sim}{W}$$
$$= (\underset{\sim}{W} \cdot \underset{\sim}{i})\,\underset{\sim}{i} \quad \text{OR} \quad (|\underset{\sim}{W}| \cos 60°)\,\underset{\sim}{i}$$
$$= 4\underset{\sim}{i},$$

Hence the force down the plane is 4 kg weight.

b In newtons, this is about $4 \times 9.8 = 39.2$ N.

c The parcel is not accelerating (it is not even moving), so the frictional force is equal and opposite to the component of the weight acting down the plane. Hence the frictional force is 4 kg weight up the plane.