INDUCTION - WHEN STEP 2 WORKS, BUT NOT STEP 1

1 Let $S(n)$ be the statement: $n^2 + n$ is an odd integer.

(a) Show that if $S(k)$ is true, then $(k+1)$ is true. **(b)** Is $S(1)$ true? **(c)** Is $S(n)$ true for any n?

(d) If the statement is not true, what change do you need to make to make it true? Prove your new statement.

a) Assume that $S(k)$ is true. , i.e $\exists q \in \mathbb{Z}$ such that $k^2 + k = 2q + 1$

$S(k+1)$ is : $(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1$

$$= \underbrace{k^2 + k}_{\text{odd by assumption}} + \underbrace{2k + 2}_{\text{even as it factorises as } 2(k+1).}$$

So $S(k+1)$ is an odd integer (being the sum of an even and an odd integer).

So if $S(k)$ is true, then $S(k+1)$ is true.

b) For $n = 1$ $n^2 + n = 1^2 + 1 = 2$, i.e. an even integer.

So $S(1)$ is not true

c) So as it's not true for $n = 1$, it's not true for any n.

d) Assume $n^2 + n$ is an even integer.

Then if we assume $S(k)$ true.

then $(k+1)^2 + (k+1) = \underbrace{k^2 + k}_{\text{even by assumption}} + \underbrace{2k + 2}_{\text{even as 2 is a factor}}$

So $S(k+1)$ is then true.

So as it's true for $n = 1$, it must be true for any n, by induction.

INDUCTION - WHEN STEP 2 WORKS, BUT NOT STEP 1

2 Given $1^2+4^2+7^2+\ldots+(3n-2)^2=\frac{n}{2}(6n^2-3n-1)$.

(a) Show that if $S(k)$ is true then $S(k+1)$ is true. **(b)** Is $S(1)$ true? **(c)** Is $S(n)$ true for any n?

a) if $S(k)$ true.

then $1^2+4^2+7^2+\ldots+(3k-2)^2+(3(k+1)-2)^2=\frac{k}{2}[6k^2-3k-1]+(3k+1)^2$

$\_\_\_\_ = \frac{k}{2}[6k^2-3k-1]+9k^2+6k+1$

$\_\_\_\_ = \frac{k(6k^2-3k-1)+18k^2+12k+2}{2}$

$\_\_\_\_ = \frac{6k^3+15k^2+11k+2}{2}$

We note that (-1) is a root of the numerator, therefore we can factorise the numerator by $(k+1)$

$\_\_\_\_ = \frac{(k+1)}{2}[6k^2+9k+2]$

$\_\_\_\_ = \frac{(k+1)}{2}[6(k+1)^2-12k-6+9k+2]$

$\_\_\_\_ = \frac{(k+1)}{2}[6(k+1)^2-3k-4]$

$\_\_\_\_ = \frac{(k+1)}{2}[6(k+1)^2-3(k+1)+3-4]$

$\_\_\_\_ = \frac{(k+1)}{2}[6(k+1)^2-3(k+1)-1]$

$\therefore$ if $S(k)$ true, then $S(k+1)$ is true.

b) For $n=1$ $\quad \frac{1}{2}(6\times1^2-3\times1-1)=\frac{2}{2}=1$ So true for $S(1)$

c) $\therefore$ by induction $S(k)$ is true for any k.

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3 It is stated that $n^2 - n + 41$ is prime for $n \geq 1$.

(a) Is this statement true for $n = 1$? **(b)** Is this statement true for $n = 2$?
(c) Is this statement true for $n = 5$?
(d) Is it possible to find a value of n for which this expression does not give a prime number? Justify your answer.
(e) Is the given statement true or false?

a) $n = 1 \Rightarrow 1^2 - 1 + 41 = 41$ which is a prime indeed.

b) for $n = 2$ $2^2 - 2 + 41 = 4 - 2 + 41 = 43$ also a prime.

c) for $n = 5$ $5^2 - 5 + 41 = 61$ which is also a prime.

d) for $n = 41$ $41^2 - 41 + 41 = 41^2 = 1681$ not a prime. as 41 is a factor

e) The statement is FALSE as it's not true for $n = 41$.

INDUCTION - WHEN STEP 2 WORKS BUT NOT STEP 1

4 Let $S(n)$ be the statement: $n^2 - n$ is an odd integer.

(a) Show that if $S(k)$ is true then $S(k+1)$ is true.

(b) Is $S(1)$ true?

(c) Is $S(n)$ true for any n?

a) Assume $S(k)$ is true. so $\exists q \in \mathbb{Z}$ such that $k^2 - k = 2q + 1$

Then $(k+1)^2 - (k+1) = k^2 + 2k + 1 - k - 1$

$= \underbrace{k^2 - k}_{\text{odd as we have assume } S(k) \text{ to be true}} + 2k$ even.

So $(k+1)^2 - (k+1)$ would be odd (being the sum of an odd and even number). $\therefore S(k+1)$ is true if $S(k)$ true.

b) For $n = 1$ $\quad 1^2 - 1 = 0$ not odd.

c) As $S(1)$ is not true, then $S(n)$ is not true.