SQUARE ROOT FUNCTIONS

Given the graph of $y = f(x)$, it is often useful or necessary to draw the graph of $y = \sqrt{f(x)}$ and the graph of $y^2 = f(x)$.

To graph these functions, it is important to find out where $f(x) < 0$, as $\sqrt{f(x)}$ is not defined for these values of x (because the square root of a negative number is not a real number). Similarly, $y^2 = f(x)$ will not be defined for these values of x, because y^2 cannot be negative.

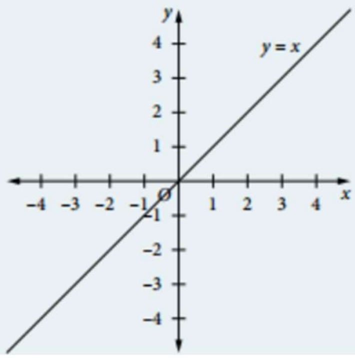
Remember that $\sqrt{0} = 0$ and $\sqrt{1} = 1$. Therefore, for $0 < f(x) < 1$, $\sqrt{f(x)} > f(x)$; for $f(x) > 1$, $f(x) > \sqrt{f(x)}$.

Graphically this means that for $0 < f(x) < 1$, the graph of $y = \sqrt{f(x)}$ is above $y = f(x)$, and for $f(x) > 1$, the graph of $y = \sqrt{f(x)}$ is below $y = f(x)$.

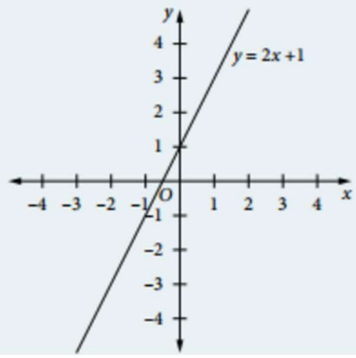
Example 4

In each part, use the graph of the given function to draw the graphs of $y = \sqrt{f(x)}$ and $y^2 = f(x)$.

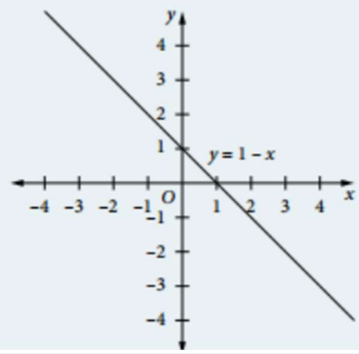
(a) Given $y = x$, draw $y = \sqrt{x}$ and $y^2 = x$.



(b) Given $y = 2x + 1$, draw $y = \sqrt{2x+1}$ and $y^2 = 2x + 1$.

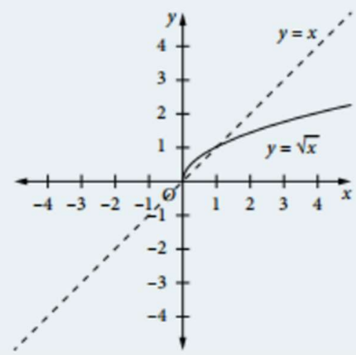


(c) Given $y = 1 - x$, draw $y = \sqrt{1-x}$ and $y^2 = 1 - x$.

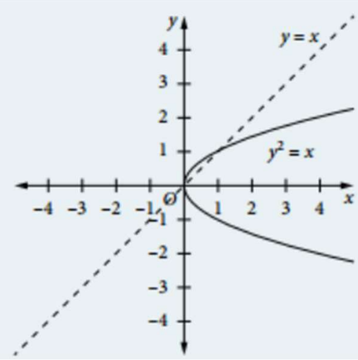


Solution

(a) The graph of $y = \sqrt{x}$ is undefined for $x < 0$.
$x = \sqrt{x}$ at $x = 0, 1$. Graphs intersect at (0, 0) and (1, 1).
For $0 < x < 1$, the graph of $y = \sqrt{x}$ is above the graph of $y = x$.



The graph of $y^2 = x$ is undefined for $x < 0$ as y^2 cannot be negative.
It has two branches, $y = \sqrt{x}$ and $y = -\sqrt{x}$.
$y = x$ and $y^2 = x$ intersect at (0, 0) and (1, 1).

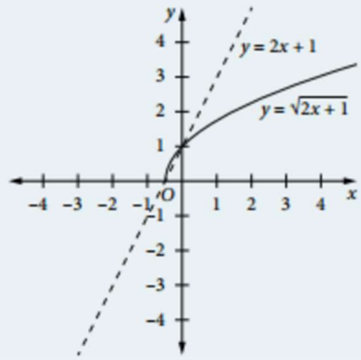


SQUARE ROOT FUNCTIONS

(b) The graph of $y = \sqrt{2x+1}$ is undefined for $x < -\frac{1}{2}$.

$2x + 1 = \sqrt{2x+1}$ at $x = -\frac{1}{2}, 0$. Graphs intersect at $\left(-\frac{1}{2}, 0\right)$ and $(0, 1)$.

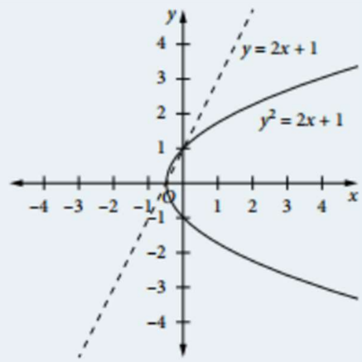
For $-\frac{1}{2} < x < 0$, the graph of $y = \sqrt{2x+1}$ is above the graph of $y = 2x + 1$.



The graph of $y^2 = 2x + 1$ is undefined for $x < -\frac{1}{2}$.

It has two branches, $y = \sqrt{2x+1}$ and $y = -\sqrt{2x+1}$.

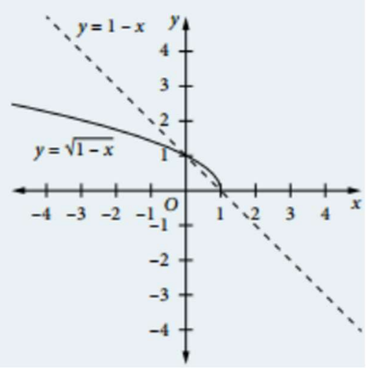
Graphs intersect at $\left(-\frac{1}{2}, 0\right)$ and $(0, 1)$.



(c) The graph of $y = \sqrt{1-x}$ is undefined for $x > 1$.

$1 - x = \sqrt{1-x}$ at $x = 0, 1$. Graphs intersect at $(0, 1)$ and $(1, 0)$.

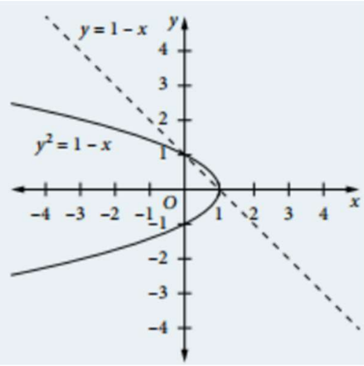
For $0 < x < 1$, the graph of $y = \sqrt{1-x}$ is above the graph of $y = 1 - x$.



The graph of $y^2 = 1 - x$ is undefined for $x > 1$.

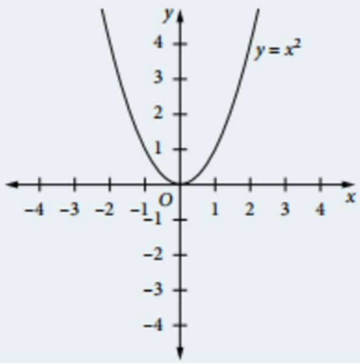
It has two branches, $y = \sqrt{1-x}$ and $y = -\sqrt{1-x}$.

Graphs intersect at $(0, 1)$ and $(1, 0)$.

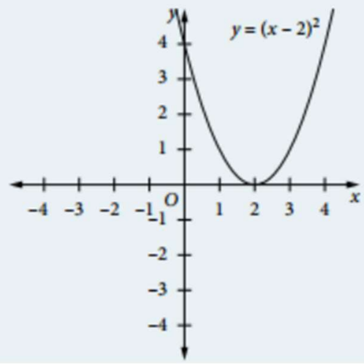


Example 5

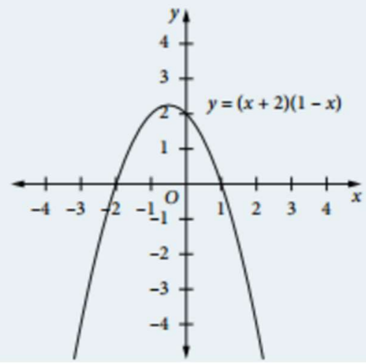
(a) Given $y = x^2$, draw $y = \sqrt{x^2}$ and $y^2 = x^2$.



(b) Given $y = (x-2)^2$, draw $y = \sqrt{(x-2)^2}$ and $y^2 = (x-2)^2$.

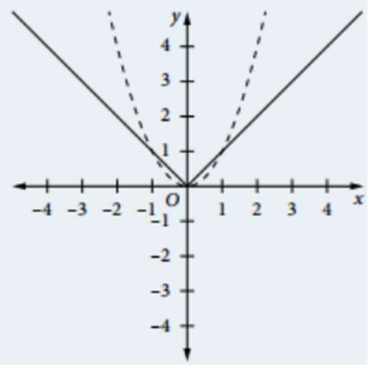


(c) Given $y = (x+2)(1-x)$, draw $y = \sqrt{(x+2)(1-x)}$ and $y^2 = (x+2)(1-x)$.

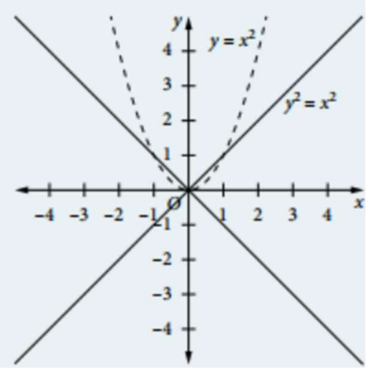


Solution

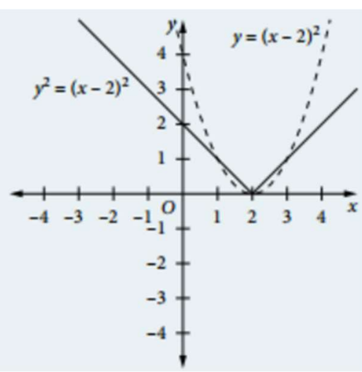
(a) The graph of $y = \sqrt{x^2}$ is defined for all x.
The function is never negative.
The graphs intersect at $(-1, 1)$, $(0, 0)$ and $(1, 1)$.
The resulting graph is the same as $y = |x|$.



The graph of $y^2 = x^2$ is defined for all x.
The graphs intersect at $(-1, 1)$, $(0, 0)$ and $(1, 1)$.
The resulting graph is the same as the graph of $y = \pm x$.



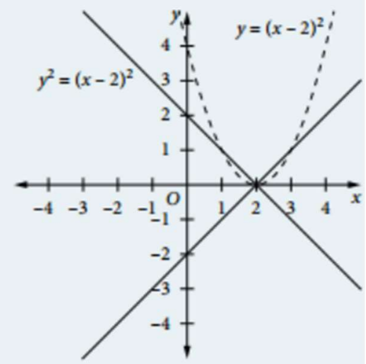
(b) The graph of $y = \sqrt{(x-2)^2}$ is defined for all x.
The function is never negative.
The graphs intersect at $(1, 1)$, $(2, 0)$ and $(3, 1)$.
The resulting graph is the same as the graph of $y = |x-2|$.



The graph of $y^2 = (x - 2)^2$ is defined for all x.

The graphs intersect at (1, 1), (2, 0) and (3, 1).

The resulting graph is the same as the graph of $y = \pm(x - 2)$.



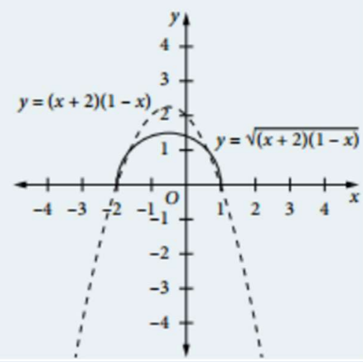
(c) The graph of $y = \sqrt{(x+2)(1-x)}$ is undefined for $x < -2$ and $x > 1$.

The maximum value of $(x+2)(1-x)$ is $\frac{9}{4}$ and occurs at $x = -\frac{1}{2}$.

The greatest value of $\sqrt{(x+2)(1-x)}$ is $\frac{3}{2}$ and occurs at $x = -\frac{1}{2}$.

$-2 < x < 1$: $0 \le y \le 1.5$.

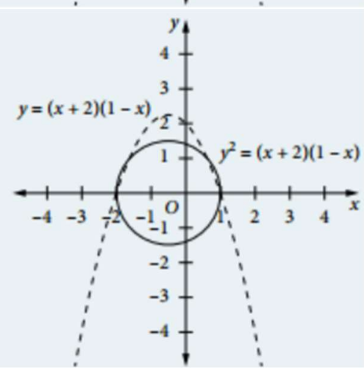
Further algebraic investigation shows that the graph is a semicircle in the upper half plane, centre (–0.5, 0), radius 1.5.



The graph of $y^2 = (x+2)(1-x)$ is undefined for $x < -2$ and $x > 1$.

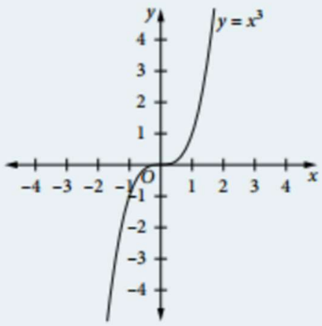
$-2 < x < 1$: $-1.5 \le y \le 1.5$.

Further algebraic investigation shows that the graph is a circle, centre (–0.5, 0), radius 1.5.

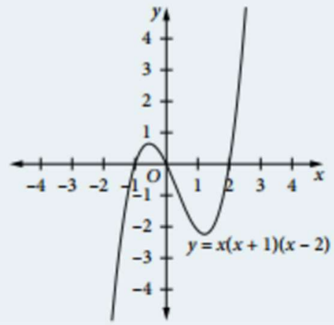


Example 6

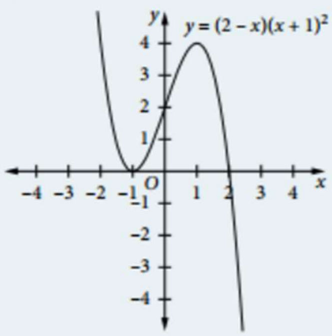
(a) Given $y = x^3$, draw $y = \sqrt{x^3}$ and $y^2 = x^3$.



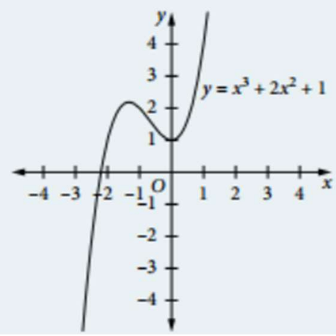
(b) Given $y = x(x+1)(x-2)$, draw $y = \sqrt{x(x+1)(x-2)}$ and $y^2 = x(x+1)(x-2)$.



(c) Given $y = (2-x)(x+1)^2$, draw $y = \sqrt{(2-x)(x+1)^2}$ and $y^2 = (2-x)(x+1)^2$.



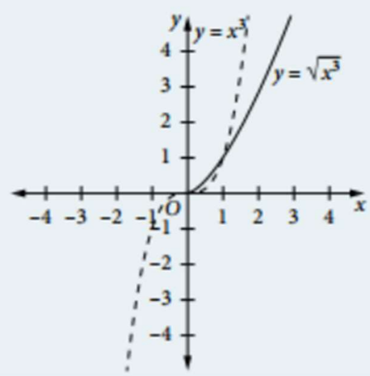
(d) Given $y = x^3 + 2x^2 + 1$, draw $y = \sqrt{x^3 + 2x^2 + 1}$ and $y^2 = x^3 + 2x^2 + 1$.



Solution

(a) The graph of $y = \sqrt{x^3}$ is undefined for $x < 0$.

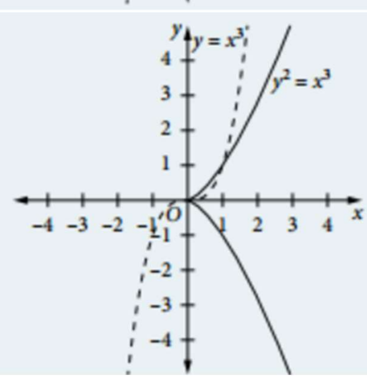
$x^3 = \sqrt{x^3}$ at $x = 0, 1$. The curves intersect at (0, 0) and (1, 1).



The graph of $y^2 = x^3$ is undefined for $x < 0$.

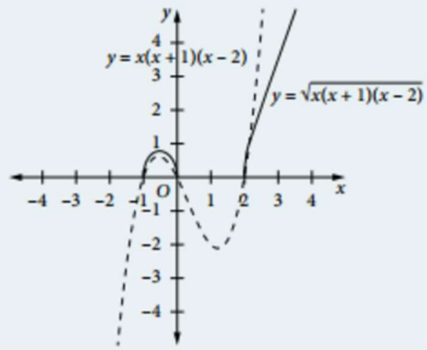
The curves intersect at (0, 0) and (1, 1).

$y^2 = x^3$ is equivalent to $y = \pm\sqrt{x^3}$.

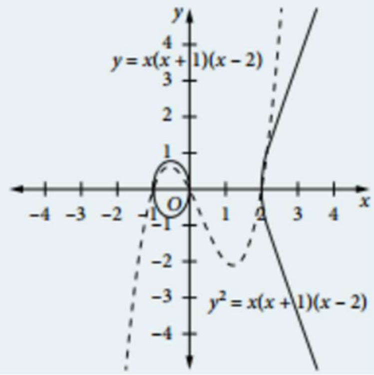


(b) The graph of $y = \sqrt{x(x+1)(x-2)}$ is undefined for $x < -1$, $1 < x < 2$.
The graphs meet at $(-1, 0)$, $(0, 0)$, $(2, 0)$.

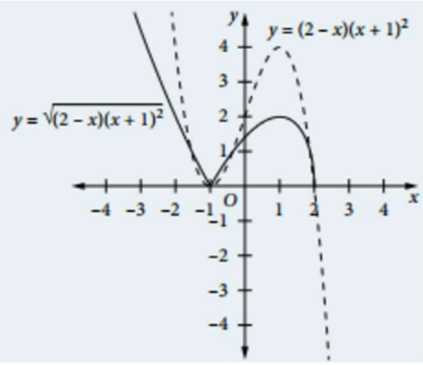
For $x > 2$, $\sqrt{x(x+1)(x-2)} > x(x+1)(x-2)$ until the RHS becomes greater than 1.



The graph of $y^2 = x(x+1)(x-2)$ is undefined for $x < -1$, $1 < x < 2$.
The graphs meet at $(-1, 0)$, $(0, 0)$, $(2, 0)$.



(c) The graph of $y = \sqrt{(2-x)(x+1)^2}$ is undefined for $x > 2$.
The maximum turning point of $y = (2-x)(x+1)^2$ is $(1, 4)$, so the maximum turning point of $y = \sqrt{(2-x)(x+1)^2}$ is $(1, 2)$.
$(-1, 0)$ is not called a turning point of $y = \sqrt{(2-x)(x+1)^2}$ because at this point the curve changes sharply, not smoothly. Instead this point is called a **cusp**.



The graph of $y^2 = (2-x)(x+1)^2$ is undefined for $x > 2$.
The maximum turning point of $y = (2-x)(x+1)^2$ is $(1, 4)$, so the maximum turning point of $y^2 = (2-x)(x+1)^2$ is $(1, 2)$ and the minimum turning point is $(1, -2)$.