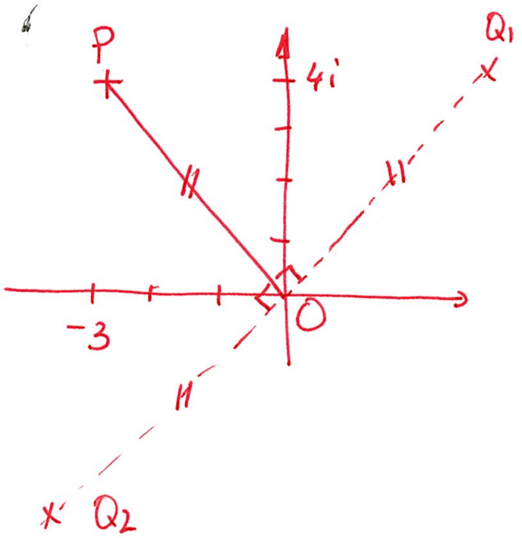
GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER AS A VECTOR

1 On an Argand diagram, point A represents the complex number α. Point B is located so that the vector $\overrightarrow{OB}$ is the result of rotating $\overrightarrow{OA}$ anticlockwise by $\frac{2\pi}{3}$ and then halving its length. Which complex number represents point B?

A $\frac{\pi}{3}\alpha$ **B** $\alpha\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ **C** $\alpha\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ $\boxed{\textbf{D}}$ $\frac{\alpha}{2}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

2 On a complex plane, P represents $z=-3+4i$ and Q represents the complex number w. Find w so that triangle OPQ is:

(a) an isosceles right-angled triangle with the right angle at O
(b) an isosceles right-angled triangle with the right angle at P
(c) right-angled at O, with OQ twice the length of OP.

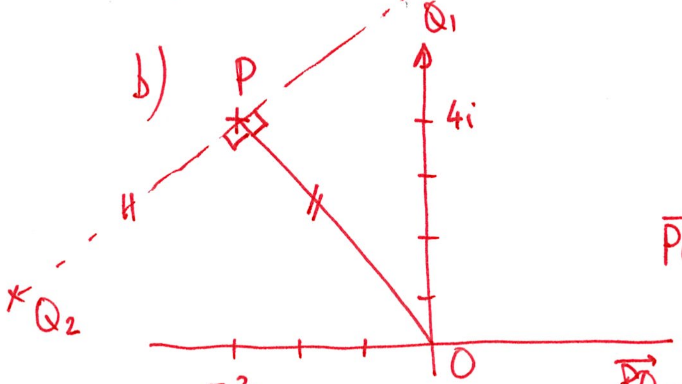


a) To get the coordinates of Q_1, we times z by $(-i)$

$w_1 = (-i)z = (-i)(-3+4i) = +4+3i$

For Q_2 $\quad w_2 = iz = i(-3+4i) = -4-3i$

b)



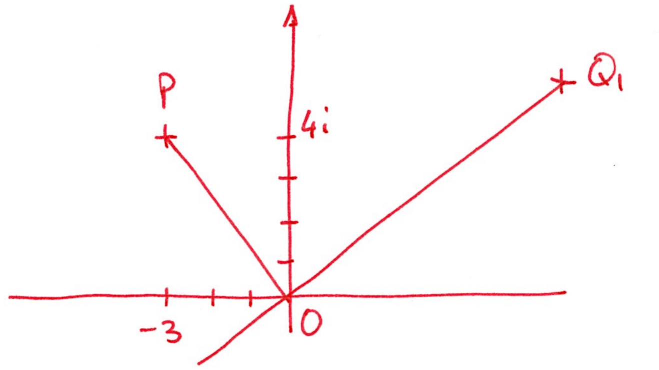
$\overrightarrow{PO} = 3-4i$ we rotate $\overrightarrow{PO}$ by $\frac{\pi}{2}$ anticlockwise so we time by i

$\overrightarrow{PQ_1} = i \times \overrightarrow{PO} = i \times (3-4i) = 4+3i = \overrightarrow{PQ_1}$

then $\overrightarrow{OQ_1} = \overrightarrow{OP} + \overrightarrow{PQ_1} = -(3-4i)+(4+3i) = 1+7i$ so $w_1 = 1+7i$

For Q_2 $\quad \overrightarrow{OQ_2} = \overrightarrow{OP} + \overrightarrow{PQ_2} = -(3-4i)+(-i)\times\overrightarrow{PO} = -3+4i-i(3-4i)$

$\overrightarrow{OQ_2} = -7+i$



c) For Q_1 $\quad \overrightarrow{OQ_1} = -2i \times z$

$\overrightarrow{OQ_1} = -2i\times(-3+4i) = 8+6i$

For Q_2

$\overrightarrow{OQ_2} = 2i\times(-3+4i) = -8-6i$

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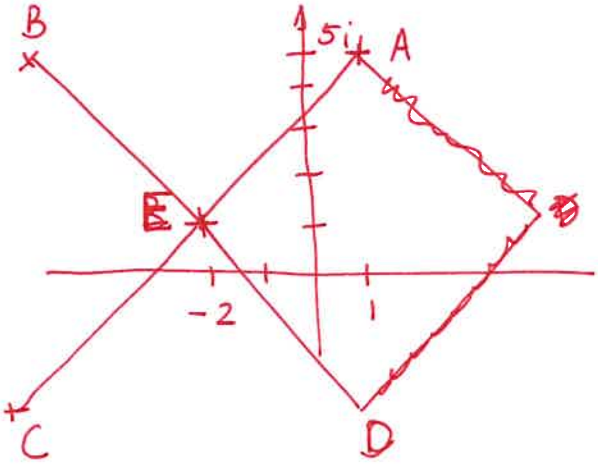
3 Point E is the centre of a square $ABCD$ (labelled anticlockwise) on an Argand diagram. E and A are the points corresponding to $-2+i$ and $1+5i$ respectively. Find the complex numbers represented by the points B, C and D.



E is centre of AC

So $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}$ or $\overrightarrow{AC} = 2\overrightarrow{AE}$

$Z_C - Z_A = 2(Z_E - Z_A)$

$Z_C = 2Z_E - 2Z_A + Z_A$

$Z_C = 2Z_E - Z_A$

$Z_C = 2(-2+i) - (1+5i) = -4 - 1 + 2i - 5i = -5 - 3i$

For B $\quad \overrightarrow{OB} = \overrightarrow{OE} + \overrightarrow{EB} = \overrightarrow{OE} + i \times \overrightarrow{EA}$

So $\quad Z_B = Z_E + i \times (Z_A - Z_E)$

$Z_B = (-2+i) + i \times [1+5i - (-2+i)]$

$Z_B = -2 + i + i \times [3+4i]$

$Z_B = -2 + i + 3i - 4$

$Z_B = -6 + 4i$

For D $\quad$ E is the centre of BD

So $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BD}$ or $\overrightarrow{BD} = 2 \times \overrightarrow{BE}$

$Z_D - Z_B = 2(Z_E - Z_B)$

$Z_D = 2Z_E - 2Z_B + Z_B = 2Z_E - Z_B$

$Z_D = 2(-2+i) - (-6+4i)$

$Z_D = -4 + 6 + 2i - 4i$

$Z_D = +2 - 2i$

4 **(a)** If $z_1 = 6 + 8i$ and $|z_2| = 15$, show that the greatest possible value of $|z_1 + z_2|$ is 25.
(b) If $|z_1 + z_2|$ takes this greatest value, find z_2 in Cartesian form.

a) From the triangle inequality: $|z_1 + z_2| \leqslant |z_1| + |z_2|$

$|z_1| = \sqrt{6^2 + 8^2} = \sqrt{36+64} = \sqrt{100} = 10$

So $|z_1 + z_2| \leqslant 10 + 15$

$|z_1 + z_2| \leqslant 25$

b) if $|z_1 + z_2| = 25 = |z_1| + |z_2|$

that means that the points represented by z_1 and z_2 and O are colinear, i.e: $z_2 = k \times z_1$ with k real.

So $|z_2| = k \times |z_1|$

$15 = k \times 10$ so $k = 15/10 = 3/2$

So $z_2 = \frac{3}{2} \times (6 + 8i) = \frac{18}{2} + \frac{24}{2} i$

$z_2 = 9 + 12i$

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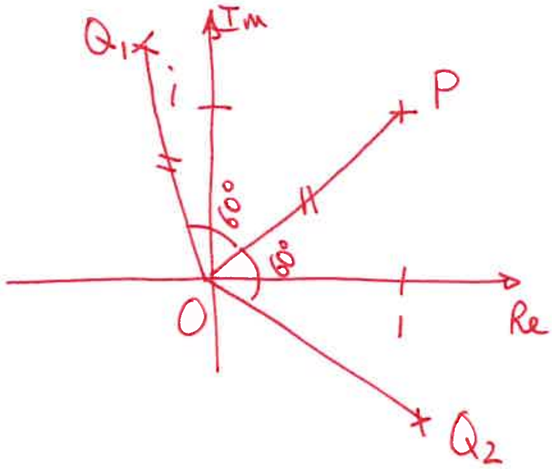
5 On an Argand diagram, P represents $z = 1 + i$ and Q represents q. Find the two possible values of q (in mod–arg form) such that ΔOPQ is equilateral.



For Q_1 : we rotate P by $\frac{\pi}{3}$ anticlockwise,

so $Z_{Q_1} = e^{i\pi/3} \times Z_P$

$Z_{Q_1} = e^{i\pi/3} \times [1+i]$

$Z_{Q_1} = [\cos \pi/3 + i \sin \frac{\pi}{3}] \times [1+i]$

$Z_{Q_1} = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \times (1+i) = \frac{1}{2} - \frac{\sqrt{3}}{2} + i\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$

$Z_{Q_1} = \left[\frac{1-\sqrt{3}}{2}\right] + i\left[\frac{1+\sqrt{3}}{2}\right]$ 1st possible value of q.

For Q_2, we rotate P by $\frac{\pi}{3}$ clockwise

so $Z_{Q_2} = e^{-i\pi/3} \times Z_P$

$Z_{Q_2} = \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right] \times (1+i)$

$Z_{Q_2} = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)(1+i)$

$Z_{Q_2} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) + i\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) = \left(\frac{1+\sqrt{3}}{2}\right) + i\left(\frac{1-\sqrt{3}}{2}\right)$

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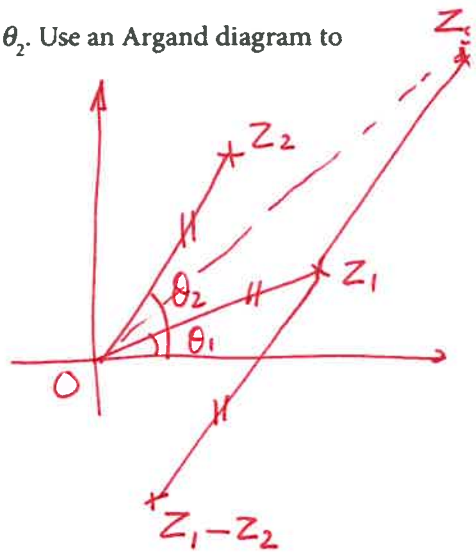
8 z_1 and z_2 are two complex numbers of equal moduli, with $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$. Use an Argand diagram to find the values of $\arg(z_1 + z_2)$ and $\arg(z_1 - z_2)$ in terms of θ_1 and θ_2.



$|z_1| = |z_2|$

$z_1 = |z_1| e^{i\theta_1}$ and $z_2 = |z_2| e^{i\theta_2} = |z_1| e^{i\theta_2}$

$z_s = z_1 + z_2$ so $\arg(z_1 + z_2) = \dfrac{\theta_1 + \theta_2}{2}$

$z_1 - z_2 = |z_1| e^{i\theta_1} - |z_1| e^{i\theta_2} = |z_1| (e^{i\theta_1} - e^{i\theta_2})$

So $\arg(z_1 - z_2) = \arg(e^{i\theta_1} - e^{i\theta_2})$

$\qquad = \arg\left[(\cos\theta_1 - \cos\theta_2) + i(\sin\theta_1 - \sin\theta_2)\right]$

$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\qquad = \arg\left[-2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right) + i\, 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}\right)\right]$

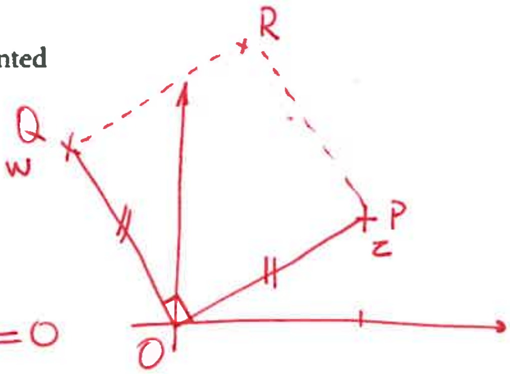
$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$

$\qquad = \arg\left[-\sin\left(\frac{\theta_1+\theta_2}{2}\right) + i\cos\left(\frac{\theta_1+\theta_2}{2}\right)\right] = \arg\left[i\left(\cos\left(\frac{\theta_1+\theta_2}{2}\right) + i\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right)\right] = \frac{\theta_1+\theta_2}{2} + \frac{\pi}{2}$

9 The points P and Q in the complex plane correspond to the complex numbers z and w respectively. Triangle OPQ is right-angled and isosceles with $OP = OQ$.

(a) Show that $w^2 + z^2 = 0$.

(b) If $OPRQ$ is a square, find (in terms of z) the complex number represented by E, the point of intersection of the diagonals of the square.



a) As OPQ is isosceles and right angled at O, we must have $w = i \times z$

So $w^2 = (iz)^2 = -z^2$ so $w^2 + z^2 = 0$

b) E is the intersection of the diagonals of the square

therefore $z_E = \dfrac{z_Q + z_P}{2} = \dfrac{w + z}{2}$

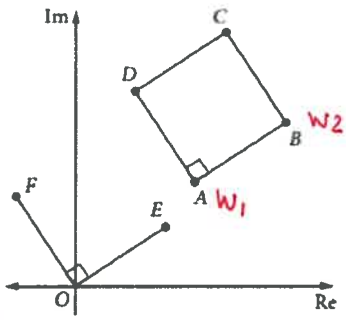
So $z_E = \dfrac{1}{2}(w + z) = \dfrac{1}{2}(iz + z) = \left(\dfrac{1+i}{2}\right) \times z$

10 On an Argand diagram, $ABCD$ is a square. OE and OF are parallel to and equal in length to AB and AD respectively. The vertices A and B correspond to the complex numbers w_1 and w_2 respectively.

(a) Explain why the point E corresponds to $w_2 - w_1$.

(b) What complex number corresponds to the point F?

(c) What complex number corresponds to the vertex D?



a) $\overrightarrow{OE} = \overrightarrow{AB}$ so $Z_E = Z_B - Z_A = W_2 - W_1$

b) F is the point E rotated $\frac{\pi}{2}$ anticlockwise.

so $Z_F = i \times Z_E = i \times (W_2 - W_1)$

c) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \overrightarrow{OA} + \overrightarrow{OF}$

So $Z_D = W_1 + i \times (W_2 - W_1)$

11 z_1 and z_2 are two complex numbers such that $\dfrac{z_1 + z_2}{z_1 - z_2} = 2i$.

(a) Show that $|z_1| = |z_2|$.

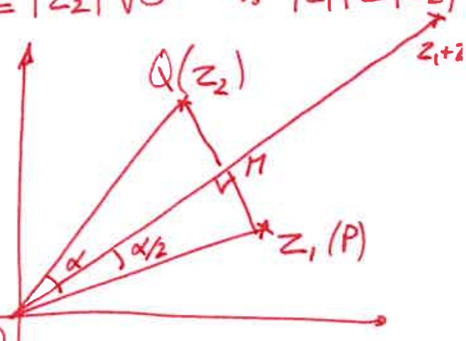
(b) If α is the angle between the vectors representing z_1 and z_2, show that $\tan\frac{\alpha}{2} = \frac{1}{2}$

(c) Show that $z_2 = \frac{1}{5}(3 + 4i)z_1$.

a) $z_1 + z_2 = 2i \times (z_1 - z_2)$ or $z_1(1-2i) = z_2(-1-2i)$

So $|z_1(1-2i)| = |z_2(-1-2i)|$ or $|z_1||1-2i| = |z_2||-1-2i|$

$|z_1|\sqrt{1^2+(-2)^2} = |z_2|\sqrt{(-1)^2+(-2)^2}$ so $|z_1|\sqrt{5} = |z_2|\sqrt{5}$ $\therefore |z_1| = |z_2|$



b) $\alpha = \theta_2 - \theta_1$

$\tan\frac{\alpha}{2} = \dfrac{MP}{OM}$ $\qquad \overrightarrow{PM} = \frac{1}{2}\overrightarrow{PQ} = \frac{1}{2}(z_2 - z_1)$

$\overrightarrow{OM} = \frac{1}{2}(z_1 + z_2)$.

So $\tan\frac{\alpha}{2} = \dfrac{\left|\frac{1}{2}(z_2 - z_1)\right|}{\left|\frac{1}{2}(z_2 + z_1)\right|} = \dfrac{|z_2 - z_1|}{|z_2 + z_1|} = \left|\dfrac{z_2 - z_1}{z_2 + z_1}\right| = \left|\dfrac{1}{2i}\right| = \dfrac{1}{2}$ so $\tan\frac{\alpha}{2} = \frac{1}{2}$

c) $z_1 + z_2 = 2i(z_1 - z_2)$ so $z_1(1-2i) = z_2(-1-2i)$

$z_2 = z_1\left(\dfrac{1-2i}{-1-2i}\right) = z_1\left(\dfrac{(1-2i)(-1+2i)}{(-1-2i)(-1+2i)}\right) = z_1\left(\dfrac{-1+4+2i(1+1)}{1+4}\right) = z_1\left(\dfrac{3+4i}{5}\right)$