SOLVING DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = g(y)$

Models that describe systems exhibiting growth or decay have the form $\frac{dy}{dx} = g(y)$. In other words, the rate of growth $\frac{dy}{dx}$ in the dependent variable y is a function $g(y)$ of the variable y alone. The function $g(y)$ on the right-hand side of such a differential equation has no explicit dependence on the independent variable x.

To solve $\frac{dy}{dx} = g(y)$ given that $y(0) = y_0$, consideration has to be given to two separate cases.

Case I: $g(y_0) = 0$

If $g(y_0) = 0$, then $y(x) = y_0$. In other words, the solution $y(x) = y_0$ is not dependent on the independent variable, so it is variously called the stationary solution, the steady state solution, or the equilibrium solution, y_s.

Case II: $g(y_0) \neq 0$

The model $\frac{dy}{dx} = g(y)$ can be transformed to a directly integrable form $\frac{1}{\left(\frac{dy}{dx}\right)} \equiv \frac{dx}{dy} = \frac{1}{g(y)}$ by taking the reciprocal of both sides, wherever $g(y_0) \neq 0$.

The solution of this new equation gives $x(y)$, so the variable x is now a function of y. In other words, the roles of the dependent and independent variables in the original equation $\frac{dy}{dx} = g(y)$ have been exchanged in the transformed equation $\frac{dx}{dy} = \frac{1}{g(y)}$.

Recall that exchanging the roles of dependent and independent variables in any relationship, such as $y = f(x)$, requires the existence of an appropriate inverse function f^{-1}. In other words, $y = f(x) \Leftrightarrow x = f^{-1}(y)$, provided f is a one-to-one function on an interval, which requires $f(x) \neq 0$ inside that interval. In this problem, $\frac{dy}{dx} = g(y)$ defines y as an invertible (i.e. one-to-one) function of x on an interval, wherever $g(y) \neq 0$ for any value of the variable y in the said solution. Therefore, the solution $x = g(y)$ of $\frac{dx}{dy} = \frac{1}{g(y)}$ can always be inverted to give the solution $y = g^{-1}(x)$ of the original problem $\frac{dy}{dx} = g(y)$ on any interval of y where $g(y) \neq 0$.

A number of these kinds of differential equations will be investigated.

To find the particular solution of $\frac{dy}{dx} = g(y)$, $y(a) = y_a$, $g(y_a) \neq 0$, requires the following five-step procedure:

- Take the reciprocal of both sides of $\frac{dy}{dx} = g(y)$ to obtain $\frac{dx}{dy} = \frac{1}{g(y)}$ with $x = a$ where $y = y_a$.
- Integrate both sides of the equation with respect to the new independent variable y:

  $$\int \frac{dx}{dy} dy = \int \frac{1}{g(y)} dy$$

  $$x + c = G(y) \qquad \text{where } G'(y) = \frac{1}{g(y)} \text{ and } c \text{ is a constant of integration.}$$
- Where possible, invert the equation from previous step to find the general solution for the original dependent variable y in terms of the original independent variable x:

  $x + c = G(y)$

  $\therefore y = G^{-1}(x + c)$
- Substitute the initial condition $(x, y) = (a, y_a)$ into the equation obtained in the previous step to evaluate the constant of integration c: $\therefore y_a = G^{-1}(a + c)$.
- Substitute the value of the constant into the general solution obtained in the previous step to obtain the particular solution.

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Example 16

Find the solution of $\frac{dy}{dx} = 2y$, given that where $x = 0$, $y = 3$.

Solution

- Take the reciprocal of both sides of the equation:
  $\frac{dx}{dy} = \frac{1}{2y}$
- Integrate with respect to y:
  $x + c = \frac{1}{2}\log_e|y|$
  Logarithms are only defined for positive quantities, so the absolute value sign is used.
- Substitute known values of x and y, $x = 0, y = 3$:
  $0 + c = \frac{1}{2}\log_e 3$
- Simplify and express with y as a function of x:
  $x + \frac{1}{2}\log_e 3 = \frac{1}{2}\log_e|y|$
  $2x = \log_e|y| - \log_e 3$
  $2x = \log_e \frac{|y|}{3}$
  $\frac{|y|}{3} = e^{2x}$
  $|y| = 3e^{2x}$
  This gives two solutions.
  If $y > 0$, $|y| = y = 3e^{2x}$
  If $y < 0$, $|y| = -y = 3e^{2x}$ $\therefore y = -3e^{2x}$
  Complete solution is: $y = \pm 3e^{2x}$

Example 17

Find the particular solution of $\frac{dy}{dt} = -r(y - s)$, given that $y(0) = y_0$ with $r, s > 0$ positive constants.

Solution

Take the reciprocal of both sides and transpose $-r$:

$-r\frac{dt}{dy} = \frac{1}{y - s}$

Integrate both sides with respect to the original dependent variable y:

$-r\int \frac{dt}{dy}dy = \int \frac{1}{y - s}dy$

$-rt + c = \log_e|y - s|$

Rearrange to find the general solution:

$|y - s| = e^c e^{-rt}$

$\therefore y - s = Ae^{-rt}$

$y = s + Ae^{-rt}$ where $A = \pm e^c$.

To find the particular solution, substitute initial condition $y(0) = y_0$:

$y_0 - s = Ae^0$

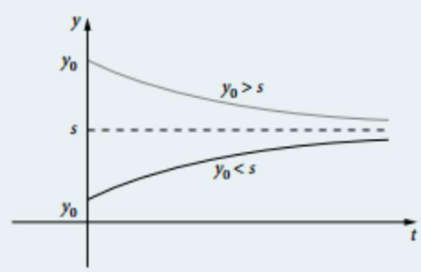
$\therefore A = y_0 - s$

$\therefore y = s + (y_0 - s)e^{-rt}$

It appears that this is a single solution. However, this solution will behave differently depending on the relative size of the initial condition $y(0) = y_0$ and the steady state solution s, as shown.

In the case that $(y_0 - s) < 0$, the solution $y = s + (y_0 - s)e^{-rt}$ grows as time increases.

In the case that $(y_0 - s) > 0$, the solution $y = s + (y_0 - s)e^{-rt}$ decays as time increases.



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Example **17** above is related to the model of uninhibited exponential growth/decay, $\frac{dy}{dt} = ry$, which for $y(0) = y_0$ has the general solution $y = y_0 e^{rx}$.

Assuming $r > 0$, the differential equation $\frac{dy}{dt} = -r(y - y_s)$, $y(0) = y_0$ is a model of inhibited growth for $y_0 < y_s$ and inhibited decay for $y_0 > y_s$. The rate of growth (or decay) in the dependent variable is called *inhibited* because the rate falls to zero as the dependent variable approaches its equilibrium value y_s.

The solution curve of an inhibited growth (or decay) problem is $y(t) = y_s + (y_0 - y_s)e^{-rt}$.

Example 18

(a) Show that $\frac{2}{(1-y)(1+y)} = \frac{1}{1-y} + \frac{1}{1+y}$.

(b) Find the general solution of $\frac{dy}{dx} = (1-y)(1+y)$, given that $y(0) = y_0$.

Solution

(a)
$$\text{RHS} = \frac{1}{1-y} + \frac{1}{1+y}$$
$$= \frac{1+y+1-y}{(1-y)(1+y)}$$
$$= \frac{2}{(1-y)(1+y)} = \text{LHS}$$

(b) Take the reciprocal of both sides of the equation:

$$\frac{dy}{dx} = (1-y)(1+y)$$
$$\frac{dx}{dy} = \frac{1}{(1-y)(1+y)}$$

Both sides of the equation are integrated with respect to y:

$$\int \frac{dx}{dy} dy = \int \frac{1}{(1-y)(1+y)} dy$$

The result from **(a)** is used, $\frac{1}{(1-y)(1+y)} = \frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right)$:

$$x = \frac{1}{2}\int \left(\frac{1}{1-y} + \frac{1}{1+y}\right) dy$$
$$2x = \int \left(\frac{1}{1-y} + \frac{1}{1+y}\right) dy$$
$$2x + C = \left(-\ln|1-y| + \ln|1+y|\right)$$
$$2x + C = \ln\left|\frac{1+y}{1-y}\right|$$

Note: Add the constant of integration to the side with the original independent variable.

Rearrange the equation from the previous step to find the general solution for the original dependent variable in terms of the original independent variable:

$$e^{2x+C} = \left|\frac{1+y}{1-y}\right|$$

$$\left|\frac{1+y}{1-y}\right| = e^{2x}e^{C}$$

$$\frac{1+y}{1-y} = \pm e^{2x}e^{C}$$

Let $A = \pm e^{C}$:

$$\frac{1+y}{1-y} = Ae^{2x}$$

$$1+y = Ae^{2x} - yAe^{2x}$$

$$y\left(1+Ae^{2x}\right) = Ae^{2x} - 1$$

$$y = \frac{Ae^{2x}-1}{1+Ae^{2x}}$$

The initial condition is substituted to find the particular solution, $y(0) = y_0$:

$$y_0 = \frac{A-1}{1+A}$$

$$y_0 + Ay_0 = A - 1$$

$$y_0 + 1 = A(1-y_0)$$

$$A = \frac{1+y_0}{1-y_0}$$

Hence the solution is:

$$y = \frac{\frac{1+y_0}{1-y_0}e^{2x} - 1}{1+\frac{1+y_0}{1-y_0}e^{2x}}$$

This may also be rearranged to give:

$$y = \frac{1-\frac{1-y_0}{1+y_0}e^{-2x}}{1+\frac{1-y_0}{1+y_0}e^{-2x}}$$

The solution of the quadratic growth rate model $\frac{dy}{dx} = (1-y)(1+y)$, $y(0) = y_0$ is:

$$y = \frac{1-\left(\frac{1-y_0}{1+y_0}\right)e^{-2x}}{1+\left(\frac{1-y_0}{1+y_0}\right)e^{-2x}}.$$