INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

2 Find the following.

(a) $\int \frac{dx}{\sqrt{16-x^2}}$ (b) $\int \frac{3}{9+x^2}dx$ (c) $\int \frac{dx}{\sqrt{1-x^2}}$ (d) $\int \frac{-1}{\sqrt{5-x^2}}dx$

a) $\int \frac{dx}{\sqrt{16-x^2}} = \int \frac{dx}{\sqrt{4^2-x^2}} = \sin^{-1}\frac{x}{4} + C$ with $-4 < x < 4$

b) $\int \frac{3}{9+x^2}dx = 3\int \frac{dx}{3^2+x^2} = \frac{3}{3}\tan^{-1}\frac{x}{3} + C = \tan^{-1}\frac{x}{3} + C$ for all x

c) $\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dx}{\sqrt{1^2-x^2}} = \sin^{-1}x + C$ with $-1 < x < 1$

d) $\int \frac{-1}{\sqrt{5-x^2}}dx = -\int \frac{1}{\sqrt{(\sqrt{5})^2-x^2}}dx$

$= \cos^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$ with $-\sqrt{5} < x < \sqrt{5}$

INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

2 Find (i) $\int \frac{-1}{\sqrt{6-x^2}}dx$ (j) $\int \frac{dx}{\sqrt{1-4x^2}}$ (k) $\int \frac{dx}{1+9x^2}$ (l) $\int \frac{dx}{9+16x^2}$

i) $\int \frac{-1}{\sqrt{6-x^2}}\,dx = \int \frac{-1}{\sqrt{(\sqrt{6})^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{\sqrt{6}}\right) + C$

j) $\int \frac{dx}{\sqrt{1-4x^2}} = \int \frac{dx}{2\sqrt{\left(\frac{1}{4}\right)-x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-x^2}} = \frac{1}{2}\sin^{-1}2x + C$

k) $\int \frac{dx}{1+9x^2} = \int \frac{1}{9} \times \frac{1}{\frac{1}{9}+x^2}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{1}{3}\right)^2+x^2}\,dx$

$\quad = \frac{1}{9} \times \frac{1}{1/3}\tan^{-1}(3x) + C = \frac{1}{3}\tan^{-1}(3x) + C$

l) $\int \frac{dx}{9+16x^2} = \int \frac{1}{16} \times \frac{dx}{\left[\left(\frac{9}{16}\right)+x^2\right]} = \frac{1}{16}\int \frac{dx}{\left(\frac{3}{4}\right)^2+x^2}$

$\quad = \frac{1}{16} \times \frac{1}{\frac{3}{4}}\tan^{-1}\left(\frac{x}{\frac{3}{4}}\right) + C$

$\quad = \frac{1}{16} \times \frac{4}{3}\tan^{-1}\left(\frac{4x}{3}\right) + C$

$\quad = \frac{1}{12}\tan^{-1}\left(\frac{4x}{3}\right) + C$

2 Find **(q)** $\int \frac{dx}{4+(x+5)^2}$ (let $u = x+5$) **(r)** $\int \frac{dx}{\sqrt{2-(x-3)^2}}$ (let $u = x-3$)

q) $u = x+5$ so $\frac{du}{dx} = 1$ or $du = dx$

$$\int \frac{dx}{4+(x+5)^2} = \int \frac{du}{4+u^2} = \int \frac{du}{2^2+u^2} = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+5}{2}\right) + C$$

r) $u = x-3$ so $du = dx$

$$\int \frac{dx}{\sqrt{2-(x-3)^2}} = \int \frac{du}{\sqrt{2-u^2}} = \int \frac{du}{\sqrt{(\sqrt{2})^2-u^2}}$$

$$= \sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= \sin^{-1}\left(\frac{x-3}{\sqrt{2}}\right) + C$$

INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

3 Evaluate **(k)** $\int_0^1 \left(\frac{1}{1+x^2}+\frac{x}{1+x^2}\right)dx$ **(m)** $\int_{-4}^{4} \frac{dx}{x^2+16}$ **(q)** $\int_{-1}^{1} \frac{dx}{\sqrt{2-x^2}}$

k) $\int_0^1 \frac{1}{1+x^2} + \frac{x}{1+x^2}\,dx = \left[\tan^{-1}(x) + \frac{1}{2}\ln(1+x^2)\right]_0^1$

$= \left[\tan^{-1}(1) + \frac{1}{2}\ln(1+1^2)\right] - \left[\tan^{-1}0 + \frac{1}{2}\ln(1+0^2)\right]$

$= \frac{\pi}{4} + \frac{1}{2}\ln 2$

m) $\int_{-4}^{4} \frac{dx}{4^2+x^2} = \left[\frac{1}{4} \times \tan^{-1}\left(\frac{x}{4}\right)\right]_{-4}^{4} = \frac{1}{4}\left[\tan^{-1}\left(\frac{x}{4}\right)\right]_{-4}^{4}$

$= \frac{1}{4}\left[\tan^{-1}1 - \tan^{-1}(-1)\right] = \frac{1}{4}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{8}$

q) $\int_{-1}^{1} \frac{dx}{\sqrt{2-x^2}} = \int_{-1}^{1} \frac{dx}{\sqrt{(\sqrt{2})^2-x^2}} = \left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_{-1}^{1}$

$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$= \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) - \sin^{-1}\left(\frac{-\sqrt{2}}{2}\right)$

$= \frac{\pi}{4} - \left(\frac{-\pi}{4}\right) = \frac{\pi}{2}$

6 On the same axes, sketch the graph of $y = 2\sin\frac{\pi x}{4}$ for $0 \le x \le 2$ and $x = 2\sin\frac{\pi y}{4}$ for $0 \le y \le 2$. Find the area of the region enclosed by the curves.

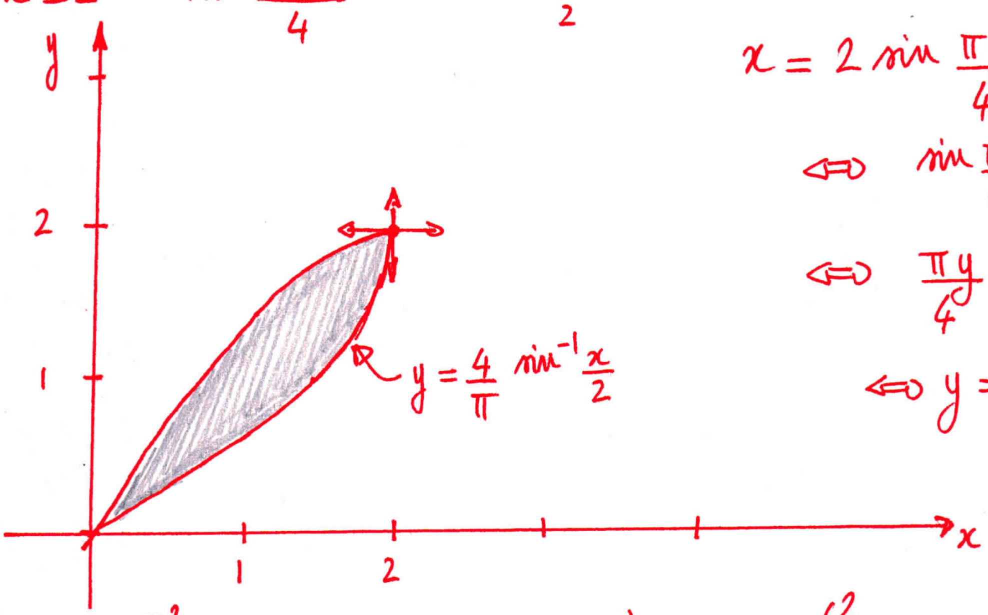
For $x = 2$ $\quad 2\sin\frac{\pi \times 2}{4} = 2 \times \sin\frac{\pi}{2} = 2$

$x = 2\sin\frac{\pi y}{4}$

$\Leftrightarrow \sin\frac{\pi y}{4} = \frac{x}{2}$

$\Leftrightarrow \frac{\pi y}{4} = \sin^{-1}\frac{x}{2}$

$\Leftrightarrow y = \frac{4}{\pi}\sin^{-1}\frac{x}{2}$

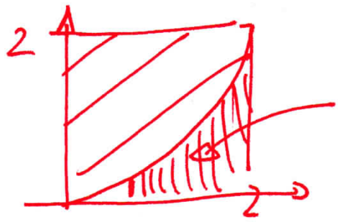


$$\text{Area} = \int_0^2 \left(2\sin\frac{\pi x}{4} - \frac{4}{\pi}\sin^{-1}\frac{x}{2}\right) dx = \int_0^2 2\sin\frac{\pi x}{4}\,dx - \frac{4}{\pi}\int_0^2 \sin^{-1}\frac{x}{2}\,dx$$

For the 1st one $\displaystyle\int_0^2 2\sin\frac{\pi x}{4}\,dx = 2\left[-\cos\frac{\pi x}{4} \times \frac{4}{\pi}\right]_0^2 = \frac{-8}{\pi}\left[\cos\frac{\pi x}{4}\right]_0^2 = \frac{8}{\pi}$

For $\displaystyle\int_0^2 \frac{4}{\pi}\sin^{-1}\frac{x}{2}\,dx$; it's equal to $2 \times 2 - \displaystyle\int_0^2 2\sin\frac{\pi x}{4}\,dx$

as



$\displaystyle\int_0^2 \frac{4}{\pi}\sin^{-1}\frac{x}{2}\,dx$ (the 2 areas form a square of side 2)

$$\text{So} \int_0^2 \frac{4}{\pi}\sin^{-1}\frac{x}{2}\,dx = 4 - 2\left[-\cos\frac{\pi x}{4} \times \frac{4}{\pi}\right]_0^2 = 4 + 2 \times \frac{4}{\pi}\left[\cos\frac{\pi x}{4}\right]_0^2$$

$$= 4 + \frac{8}{\pi}(-1) = 4 - 8/\pi$$

$$\text{So Area} = \frac{8}{\pi} - \left(4 - \frac{8}{\pi}\right) = \frac{16}{\pi} - 4$$

7 The curve $y=\frac{1}{\sqrt{1+x^2}}$ is rotated about the x-axis. Find the volume of the solid enclosed between $x=\frac{1}{\sqrt{3}}$ and $x=\sqrt{3}$.

We look for $\int_{1/\sqrt{3}}^{\sqrt{3}} \pi \times \frac{1}{\left(\sqrt{1+x^2}\right)^2}\,dx = \pi \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1+x^2}\,dx$

$= \pi \left[\tan^{-1}(x)\right]_{1/\sqrt{3}}^{\sqrt{3}}$

$= \pi \left[\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$

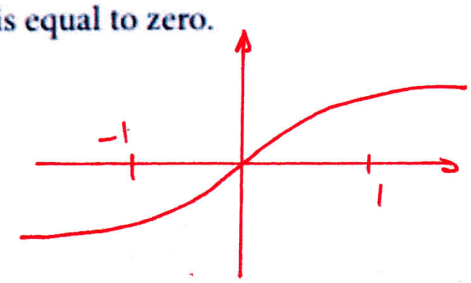
$= \pi \left[\frac{\pi}{3} - \frac{\pi}{6}\right]$

$= \pi^2 \times \frac{1}{6} = \frac{\pi^2}{6}$

9 Without evaluating the integral, explain why $\int_{-1}^{1} \tan^{-1} x\,dx$ is equal to zero.

$f(x) = \tan^{-1}x$ is an odd function,

$\therefore \forall a \in \mathbb{R} \int_{-a}^{a} \tan^{-1}x\,dx = 0$



12 Differentiate $x\cos^{-1}x - \sqrt{1-x^2}$ and use the result to evaluate $\int_0^1 \cos^{-1} x\,dx$.

let $f(x) = x\cos^{-1}x - \sqrt{1-x^2}$

$\frac{d}{dx}(f(x)) = \left[1 \times \cos^{-1}x - \frac{x}{\sqrt{1-x^2}}\right] - \frac{1}{2} \times (1-x^2)^{\frac{1}{2}-1} \times (-2x)$

so $f'(x) = \cos^{-1}x - \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} = \cos^{-1}x$

$\therefore \int_0^1 \cos^{-1}x\,dx = \int_0^1 \frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)dx$

$= \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^1$

$= \left[1\cos^{-1}1 - \sqrt{1-1^2}\right] - \left[0 - \sqrt{1-0^2}\right]$

$= 0 - 0 - 0 + 1$

$= 1$

INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

14 **(a)** Prove that $\frac{d}{dx}\left(x\sin^{-1}x\right)=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}$.

(b) Hence show that $\int_0^{\frac{1}{2}}\sin^{-1}x\,dx=\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1$. (You may use the substitution $u=1-x^2$.)

a) $\frac{d}{dx}\left(x\sin^{-1}x\right) = \sin^{-1}x + x \times \frac{1}{\sqrt{1-x^2}}$ (product rule)

b) $\int_0^{1/2}\sin^{-1}x\,dx = \int_0^{1/2}\frac{d}{dx}\left(x\sin^{-1}x\right) - \frac{x}{\sqrt{1-x^2}}\,dx$

$= \left[x\sin^{-1}x\right]_0^{1/2} - \int_0^{1/2}\frac{x}{\sqrt{1-x^2}}\,dx$

$= \frac{1}{2}\sin^{-1}\frac{1}{2} + \frac{1}{2}\int_0^{1/2}\frac{(-2x)}{\sqrt{1-x^2}}\,dx$

$= \frac{1}{2}\times\frac{\pi}{6} + \frac{1}{2}\left[\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_0^{1/2}$

$= \frac{\pi}{12} + \left[\sqrt{1-x^2}\right]_0^{1/2}$

$= \frac{\pi}{12} + \left[\sqrt{1-\frac{1}{4}} - \sqrt{1}\right]$

$= \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$