SOLVING EQUATIONS WITH LOGARITHMS

1 Solve for x:

(a) $\log_3 x = 3$

$3^3 = x$

so $x = 27$

(b) $\log_x 81 = 2$

$x^2 = 81$

$x = 9$

(c) $\log_6 x = 3$

$6^3 = x$

$x = 216$

(d) $\log_x 343 = 3$

$x^3 = 343$

$x = \sqrt[3]{343} = 7$

(e) $\log_5 x = -3$

$5^{-3} = x$

$x = \frac{1}{5^3} = \frac{1}{125}$

(f) $\log_3 81 = x$

$x = \log_3 3^4$

$x = 4 \log_3 3$

$x = 4$

(g) $\log_x \frac{1}{64} = -3$

$x^{-3} = 1/64$

so $x^3 = 64$

$x = \sqrt[3]{64} = 4$

(h) $\log_9 x = 0.25$

$9^{0.25} = x$

$x = 9^{1/4} = (3^2)^{1/4} = 3^{2/4} = 3^{1/2}$

$x = 3^{1/2} = \sqrt{3}$

(i) $\log_3 27\sqrt{3} = x$

$x = \log_3 3^3 3^{1/2}$

$x = \log_3 3^{7/2}$

so $x = \frac{7}{2} \log_3 3$

$x = \frac{7}{2}$

(j) $\log_7 x = 2.5$

$x = 7^{2.5}$

$x = 7^2 \times \sqrt{7}$

$x = 49\sqrt{7}$

(k) $\log_2 (\log_2 x) = 2$

$2^2 = \log_2 x$

so $\log_2 x = 4$

so $x = 2^4$

$x = 16$

(l) $\log_2 x = \log_2 8 + \log_4 8$

$\log_2 x = \log_2 8 + \frac{\log_2 8}{\log_2 4}$

$\log_2 x = 3 + \frac{3}{2} = \frac{9}{2}$

so $x = 2^{9/2} = 2^{4+1/2}$

$x = 16\sqrt{2}$

2 Without using a calculator, solve each equation:

(a) $\log_{10} x = \log_{10} 4 + \log_{10} 2$

$\log_{10} x = \log_{10} 4 \times 2$

so $\log_{10} x = \log_{10} 8$

$x = 8$

(b) $\log_{10} x = \log_{10} 4 - \log_{10} 2$

$\log_{10} x = \log_{10} \frac{4}{2}$

so $\log_{10} x = \log_{10} 2$

$x = 2$

(c) $\log_{10} x = \frac{\log_{10} 4}{\log_{10} 2}$

$\log_{10} x = \frac{\log_{10} 2^2}{\log_{10} 2}$

so $\log_{10} x = \frac{2 \times \log_{10} 2}{\log_{10} 2}$

so $\log_{10} x = 2$

$x = 10^2 = 100$

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(d) $\log_{10} x = \frac{1}{2}\log_{10}\left(\frac{1}{4}\right)$

$\log_{10} x = \log_{10}\left(\frac{1}{4}\right)^{1/2}$

so $x = \left(\frac{1}{4}\right)^{1/2} = \frac{1}{\sqrt{4}}$

$x = \frac{1}{2}$

(e) $2\log_{10}x + 3 = \log_{10}(x^5)$

$\Leftrightarrow 2\log_{10} x + 3 = 5\log_{10} x$

$\Leftrightarrow 3\log_{10} x = 3$

$\Leftrightarrow \log_{10} x = 1$

$x = 10$

(f) $\log_{10}x^2 = 2$

$\Leftrightarrow x^2 = 10^2 = 100$

$\Leftrightarrow x = \pm 10$

(Note that in this case, x could be positive or negative, as both are possible in the original equation)

4 Solve:

(a) $\log_{10}2 + \log_{10}5 + \log_{10}x - \log_{10}3 = 2$

$\Leftrightarrow \log_{10}\left(\frac{2\times 5\times x}{3}\right) = 2$

$\Leftrightarrow \log_{10}\left(\frac{10x}{3}\right) = 2 = \log_{10}100$

so $\frac{10x}{3} = 100$

$x = 30$

(b) $2\log_{10}x + 3 = 5\log_{10}x$

$\Leftrightarrow 2\log_{10} x + 3 - 5\log_{10} x = 0$

$\Leftrightarrow 3\log_{10} x = 3$

$\Leftrightarrow \log_{10} x = 1$

$x = 10$

(c) $\log_{10}2 + 5\log_{10}x - \log_{10}5 - \log_{10}(x^3) = \log_{10}40$

$\Leftrightarrow \log_{10}\left(\frac{2\times x^5}{5\times x^3}\right) = \log_{10}40$

$\Leftrightarrow \log_{10}\left(\frac{2x^2}{5}\right) = \log_{10}40$

so $\frac{2x^2}{5} = 40$

$x^2 = 100$

$x = 10$

(x cannot be negative as it's not possible in the original equation)

(d) $\log_{10}x = 4\log_{10}2 - 2\log_{10}x$

$3\log_{10} x = \log_{10} 2^4$

$x^3 = 2^4 = 16$

$x = \sqrt[3]{16}$

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(e) $\log_{10} x - \log_{10}(x-1) = 1$

$\Leftrightarrow \log_{10}\left(\dfrac{x}{x-1}\right) = 1 = \log_{10} 10$

so $\dfrac{x}{x-1} = 10$

$\Leftrightarrow x = 10x - 10$

$\Leftrightarrow 9x = 10$

$x = \dfrac{10}{9} = 1\frac{1}{9}$

(f) $\log_{10} x = 2\log_{10} 3 + \log_{10} 5 - \log_{10} 2 - 1$

$\log_{10} x = \log 3^2 + \log_{10} 5 - \log_{10} 2 - \log_{10} 10$

$\log_{10} x = \log_{10}\left(\dfrac{9 \times 5}{2 \times 10}\right) = \log_{10}\left(\dfrac{45}{20}\right)$

$x = \dfrac{45}{20} = \dfrac{9}{4} = 2\frac{1}{4}$

5 Solve $2^{-x} = 5$. Indicate whether each statement below is a correct or incorrect step in the solution.

(a) $x = \dfrac{\log 5}{\log 2}$ NO **(b)** $x = \log_2\left(\frac{1}{5}\right)$ Yes **(c)** $x = \dfrac{-\log 5}{\log 2}$ Yes **(d)** $x = -2.32$ (2 d.p.) Yes.

$2^{-x} = 5$ so $\log_2 2^{-x} = \log_2 5 \Leftrightarrow -x \underbrace{\log_2 2}_{=1} = \log_2 5 \Leftrightarrow x = -\log_2 5$
or $x = \log_2 5^{-1}$

or $\Leftrightarrow \log_{10} 2^{-x} = \log_{10} 5 \Leftrightarrow -x \log_{10} 2 = \log_{10} 5 \Leftrightarrow x = \dfrac{-\log_{10} 5}{\log_{10} 2}$

$x = -\dfrac{\log_{10} 5}{\log_{10} 2} \simeq 2.32$

6 Solve, correct to 3 decimal places:

(a) $2^x = 7$

$\ln 2^x = \ln 7$

$x \ln 2 = \ln 7$

so $x = \dfrac{\ln 7}{\ln 2} \simeq 2.807$

(indeed $2^{2.807} \simeq 7$)

(b) $3^x = 18$

$\ln 3^x = \ln 18$

$x \ln 3 = \ln 18$

$x = \dfrac{\ln 18}{\ln 3}$

$x \simeq 2.631$

(c) $5^x = 2$

$\ln 5^x = \ln 2$

$x \ln 5 = \ln 2$

$x = \dfrac{\ln 2}{\ln 5}$

$x \simeq 0.431$

(d) $0.4^x = 2$

$\ln 0.4^x = \ln 2$

$x = \dfrac{\ln 2}{\ln 0.4}$

$x \simeq -0.756$

(e) $6^x = 21$

$\ln 6^x = \ln 21$

$x \ln 6 = \ln 21$

$x = \dfrac{\ln 21}{\ln 6}$

$x \simeq 1.699$

(f) $3^{-x} = 0.1$

$\ln 3^{-x} = \ln 0.1$

$-x \ln 3 = \ln 0.1$

$x = -\dfrac{\ln 0.1}{\ln 3}$

$x \simeq 2.096$

(g) $5^x = 16$

$\ln 5^x = \ln 16$

$x \ln 5 = \ln 16$

$x = \dfrac{\ln 16}{\ln 5}$

$x \simeq 1.723$

(h) $4^x = 5$

$\ln 4^x = \ln 5$

$x \ln 4 = \ln 5$

$x = \dfrac{\ln 5}{\ln 4}$

$x \simeq 1.161$

7 Find the values of x (to 2 decimal places) for which:

(a) $5^x > 2$

$\ln 5^x > \ln 2$

$x \ln 5 > \ln 2$

$x > \dfrac{\ln 2}{\ln 5}$

$x > 0.43$

(b) $1.6^x \geq 0.5$

$\ln 1.6^x \geq \ln 0.5$

$x \ln 1.6 \geq \ln 0.5$

$x \geq \dfrac{\ln 0.5}{\ln 1.6}$

$x \geq -1.47$

(c) $3^x < 0.2$

$\ln 3^x < \ln 0.2$

$x \ln 3 < \ln 0.2$

$x < \dfrac{\ln 2}{\ln 3}$

$x < -1.46$

(d) $3^{-x} > 27$

$3^{-x} > 3^3$

so $-x > 3$

$x < -3$

8 If $y = a10^{bx}$, then:

A $x = \log_{10} \frac{y}{ab}$ NO

B $x = \frac{1}{b} \log_{10} \frac{y}{a}$ correct

C $y = \frac{1}{b} \log_{10} \frac{x}{a}$ NO

D $x = \frac{1}{a} \log_{10} \frac{y}{b}$ NO

$10^{bx} = \dfrac{y}{a}$ so $\log_{10} 10^{bx} = \log_{10}\left(\dfrac{y}{a}\right) \Leftrightarrow bx \times \underbrace{\log_{10} 10}_{=1} = \log_{10}\left(\dfrac{y}{a}\right)$

so $bx = \log_{10}\left(\dfrac{y}{a}\right) \Leftrightarrow x = \dfrac{1}{b} \log_{10}\left(\dfrac{y}{a}\right)$

9 If $\log_{10} A = bt + \log_{10} P$, express A in terms of the other symbols.

$A = 10^{(bt + \log_{10} P)} = 10^{bt} \times 10^{\log_{10} P} = 10^{bt} \times P$

So $A = P \times 10^{bt}$

10 If $\log y = \log a + n \log x$, find an expression for y.

$\Leftrightarrow \log y = \log a + \log x^n = \log(ax^n)$

so $y = a x^n$

11 If $y = \dfrac{\log x}{\log 2}$, express x in terms of y.

$\Leftrightarrow \log x = y \times \log 2 = \log 2^y$

so $x = 2^y$

12 If $x = a^2\sqrt{b^3 c}$, express $\log x$ in terms of $\log a$, $\log b$ and $\log c$.

$\log x = \log\left(a^2\sqrt{b^3c}\right) = \log a^2 + \log\left(\sqrt{b^3c}\right) = 2\log a + \log\left[(b^3c)^{1/2}\right]$

so $\log x = 2\log a + \frac{1}{2}\left[\log(b^3c)\right] = 2\log a + \frac{1}{2}\left[\log b^3 + \log c\right]$

so $\log x = 2\log a + \frac{3}{2}\log b + \frac{1}{2}\log c$

13 If $\log x = 0.6$ and $\log y = 0.2$, evaluate $\log\left(\frac{x^2}{\sqrt{y}}\right)$.

$$\log\left(\frac{x^2}{\sqrt{y}}\right) = \log x^2 - \log(\sqrt{y}) = 2\log x - \frac{1}{2}\log y$$

$$\text{So } \log\left(\frac{x^2}{\sqrt{y}}\right) = 2 \times 0.6 - \frac{1}{2} \times 0.2 = 1.1$$

14 If $y = ae^{4t}$, express t in terms of a and y. $\iff e^{4t} = y/a$

$$\ln\left(\frac{y}{a}\right) = \ln e^{4t} \quad \text{so} \quad 4t \underbrace{\ln e}_{=1} = \ln(y/a)$$

$$\text{so } 4t = \ln(y/a) \quad \text{so} \quad t = \frac{1}{4}\ln(y/a)$$

15 If $\log_b a = p$ and $c = a^2$, find the following in terms of p: **(a)** $\log_b c$ **(b)** $\log_c b$

a) $\log_b c = \log_b a^2 = 2\log_b a = 2p$

b) $\log_c b = \frac{\log_b b}{\log_b c} = \frac{1}{2p}$ so $\log_c b = \frac{1}{2p}$

16 If $\log_a 2 = \log_b 16$, show that $b = a^4$.

$$\log_a 2 = \log_b 16 = \log_b 2^4 = 4\log_b 2 = 4\frac{\log_a 2}{\log_a b}$$

$$\therefore 1 = \frac{4}{\log_a b} \Rightarrow \log_a b = 4 \quad \text{so} \quad a^4 = b$$

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17 \$5000 is invested at 7% p.a. compound interest. How long does it take for this money to:

(a) double in value (b) grow to \$20000 (c) grow to \$30000?

For compound interest, the formula is $P_n = 5{,}000 \times (1 + 0.07)^n$

So $P_n = 5{,}000\,(1.07)^n$

a) For P_n to be equal to 10,000, then $10{,}000 = 5{,}000 \times (1.07)^n$

$\Leftrightarrow 1.07^n = 2 \Leftrightarrow \ln 1.07^n = \ln 2 \Leftrightarrow n \times \ln 1.07 = \ln 2$

$\therefore n = \dfrac{\ln 2}{\ln 1.07} = 10.2$ years

b) P_n is equal to 20,000, so $20{,}000 = 5{,}000 \times 1.07^n$

so $1.07^n = 4$ so $n = \dfrac{\ln 4}{\ln 1.07} = 20.5$ years approx

c) $30{,}000 = 5{,}000 \times 1.07^n$

so $\ln 1.07^n = \ln \dfrac{30{,}000}{5{,}000} = \ln 6$

so $n \ln 1.07 = \ln 6$

$n = \dfrac{\ln 6}{\ln 1.07} \simeq 26.5$ years.

18 \$5000 is invested at 6% p.a. compound interest. If the interest is calculated monthly, how long does it take for this money to:

(a) double in value **(b)** grow to \$20000 **(c)** grow to \$30000?

6% p.a corresponds to $\frac{0.06}{12} = 0.005$ per month.

a) $P_n = 5{,}000 \times (1 + 0.005)^n = 5{,}000 \times 1.005^n$

So if $P_n = 10{,}000$, then $1.005^n = 2$ so $n = \frac{\ln 2}{\ln 1.005} \simeq 139$ months

so approx 11 years and 7 months.

b) $20{,}000 = 5000 \times 1.005^n$ so $n = \frac{\ln 4}{\ln 1.005} = 278$ months

So approx 23 years and 2 months

c) $30{,}000 = 5000 \times 1.005^n$ so $n = \frac{\ln 6}{\ln 1.005} = 359$ months

so approx 29 years and 11 months

19 Marika and Joe deposit \$4000 in an account that pays 9% p.a. compound interest, to be withdrawn when it has grown to \$20000. If the interest is calculated monthly, for how many whole months must they leave the money in the account?

$P_n = 4{,}000 \times \left(1 + \frac{0.09}{12}\right)^n = 4{,}000 \times 1.0075^n$

so if $P_n = 20{,}000$, then: $20{,}000 = 4{,}000 \times 1.0075^n$

so $1.0075^n = 5 \iff \ln 1.0075^n = \ln 5$

$\iff n \times \ln 1.0075 = \ln 5$

so $n = \frac{\ln 5}{\ln 1.0075} \simeq 215$ months (or 17 years and 11 months)