INTEGRATION INVOLVING LOGARITHMIC FUNCTIONS

1 Find $f(x)$ and the domain of f for the following.

(a) $f'(x)=\frac{1}{x+1}$ (b) $f'(x)=\frac{2}{2x+1}$ (c) $f'(x)=\frac{x^2}{8-x^3}$ (d) $f'(x)=\frac{1}{2-4x}$

a) $f(x)=\int \frac{1}{x+1}\,dx = \ln|x+1|+C = \ln(x+1)+C$ with $x>-1$

b) $f(x)=\int \frac{2}{2x+1}\,dx = \ln|2x+1|+C = \ln(2x+1)+C$ with $x>-1/2$

c) $f(x)=\int \frac{x^2}{8-x^3}\,dx = \left[\int \frac{-3x^2}{8-x^3}\,dx\right] \times \left(-\frac{1}{3}\right)$

$= -\frac{1}{3}\left[\ln|8-x^3|\right]+C = -\frac{1}{3}\ln(8-x^3)+C$

with $8-x^3>0 \Leftrightarrow x^3<8$
$\Leftrightarrow x<2$

d) $f(x)=\int \frac{1}{2-4x}\,dx = \left[\int \frac{-4}{2-4x}\,dx\right] \times \left(-\frac{1}{4}\right)$

$= -\frac{1}{4}\times \ln|2-4x|+C$

$= -\frac{1}{4}\times \ln(2-4x)+C$ with $2-4x>0$
$\Leftrightarrow 4x<2 \Leftrightarrow x<1/2$

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(i) $f'(x) = \dfrac{x}{(1+x^2)^2}$ (j) $f'(x) = \dfrac{x}{\sqrt{1+x^2}}$ (k) $f'(x) = \dfrac{1}{2x+5}$ (l) $f'(x) = \dfrac{1}{(2x+5)^2}$

i) $f(x) = \int \dfrac{x}{(1+x^2)^2}\,dx$ We do a change of variable $X = 1 + x^2$

$\dfrac{dX}{dx} = 2x$ so $dX = 2x\,dx$ or $x\,dx = \dfrac{1}{2}dX$

$$\int \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}\int \frac{dX}{X^2} = \frac{1}{2}\int X^{-2}\,dX$$

$$\text{———} = \frac{1}{2}\left(\frac{X^{-1}}{-1}\right) + C = -\frac{1}{2X} + C = -\frac{1}{2(1+x^2)} + C$$

j) $f(x) = \int \dfrac{x}{\sqrt{1+x^2}}\,dx = \dfrac{1}{2}\int \dfrac{(2x)}{\sqrt{1+x^2}}\,dx$

Change of variable $X = 1 + x^2$ so $\dfrac{dX}{dx} = 2x$ $dX = 2x\,dx$

$$f(x) = \frac{1}{2}\int \frac{dX}{X^{1/2}} = \frac{1}{2}\int X^{-1/2}\,dX = \left(\frac{1}{2}\right) \times \left(\frac{X^{-1/2+1}}{-1/2+1}\right) + C$$

$$f(x) = \left(\frac{1}{2}\right) \times \frac{X^{1/2}}{1/2} + C = X^{1/2} + C = \sqrt{1+x^2} + C$$

k) $f(x) = \int \dfrac{1}{2x+5}\,dx = \dfrac{1}{2}\int \dfrac{2}{2x+5}\,dx = \dfrac{1}{2}\ln|2x+5| + C$

$f(x) = \ln\sqrt{2x+5} + C$ with $2x + 5 > 0$

$\Leftrightarrow x > -5/2$

e) $f(x) = \int \frac{1}{(2x+5)^2} dx = \int (2x+5)^{-2} dx$

$$f(x) = \frac{(2x+5)^{-2+1}}{(-2+1)} \times \frac{1}{2} + C$$

$$f(x) = \frac{(2x+5)^{-1}}{(-1)} \times \frac{1}{2} + C$$

So $f(x) = -\frac{1}{2(2x+5)} + C \qquad x \neq -5/2$

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(q) $f'(x)=\dfrac{x^2-5x+1}{x-2}$ (r) $f'(x)=\dfrac{x^3}{x+1}$ (s) $f'(x)=\dfrac{x+3}{x^2+6x-7}$ (t) $f'(x)=\cot x$

q) $f(x)=\displaystyle\int \frac{x^2-5x+1}{x-2}\,dx$ The power of the numerator is greater than the power of the denominator, so we need to do a long division of the polynomials.

$$
\begin{array}{r|l}
 & x-3 \\
x-2 & x^2-5x+1 \\
 & \underline{x^2-2x} \\
 & -3x+1 \\
 & \underline{-3x+6} \\
 & -5
\end{array}
$$

so $x^2-5x+1=(x-2)(x-3)-5$

$\therefore\ f(x)=\displaystyle\int \frac{(x-2)(x-3)-5}{(x-2)}\,dx=\int\left[(x-3)-\frac{5}{(x-2)}\right]dx$

$f(x)=\dfrac{x^2}{2}-3x-5\ln|x-2|+C=\dfrac{x^2}{2}-3x-5\ln(x-2)+C$ with $x>2$

r) $f(x)=\displaystyle\int\frac{x^3}{x+1}\,dx$

$$
\begin{array}{r|l}
 & x^2-x+1 \\
x+1 & x^3 \\
 & \underline{x^3+x^2} \\
 & -x^2 \\
 & \underline{-x^2-x} \\
 & x \\
 & \underline{-x+1} \\
 & -1
\end{array}
$$

$\therefore\ x^3=(x+1)(x^2-x+1)-1$

$f(x)=\displaystyle\int\frac{(x^2-x+1)(x+1)-1}{(x+1)}\,dx$

$f(x)=\displaystyle\int x^2-x+1-\frac{1}{x+1}\,dx=\frac{x^3}{3}-\frac{x^2}{2}+x-\ln|x+1|+C$

$f(x)=\dfrac{x^3}{3}-\dfrac{x^2}{2}+x-\ln(x+1)+C$ with $x>-1$

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s) $f(x) = \int \frac{x+3}{x^2+6x-7}\, dx = \int \frac{2x+6 \ -x-3}{x^2+6x-7}\, dx$

Note: the power of the numerator is one less than the power of the denominator, so we try to go to a form $\frac{f'(x)}{f(x)}$

$$f(x) = \int \frac{2x+6}{x^2+6x-7}\, dx - \int \frac{(x+3)}{x^2+6x-7}\, dx$$

$$f(x) = \ln|x^2+6x-7| - \frac{1}{2}\int \frac{2x+6}{x^2+6x-7}\, dx$$

$$f(x) = \ln|x^2+6x-7| - \frac{1}{2}\ln|x^2+6x-7| + C$$

$$f(x) = \frac{1}{2}\ln|x^2+6x-7| + C \qquad \Delta = 36 - 4\times(-7) = 8 = (2\sqrt{2})^2$$

two roots $x_1 = \frac{-6+2\sqrt{2}}{2} = -3+\sqrt{2}$ and $x_2 = -3-\sqrt{2}$

So $f(x) = \frac{1}{2}\ln(x^2+6x-7) + C$ with $x < -3-\sqrt{2}$ or $x > -3+\sqrt{2}$

t) $f(x) = \int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx = \int \frac{(\sin x)'}{\sin x}\, dx$

$$f(x) = \ln|\sin x| + C$$

$f(x) = \ln(\sin x) + C$ with $2n\pi < x < (2n+1)\pi$

(so that $\sin x$ is always strictly positive)

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2 $\int \frac{x^3+2x^2+3x+2}{x^2+1}dx = \ldots$

A $\frac{x^2}{2}+2x+\log_e(x^2+1)+C$

B $\frac{x^2}{2}+2x+\tan^{-1}x+C$

C $\frac{x^2}{2}+2x+\log_e\sqrt{x^2+1}+C$

D $\frac{x^2}{2}+2x+\tan^{-1}\frac{x}{2}+C$

The power of the numerator is greater than the denominator therefore we need to do a long division of polynomials.

$$\begin{array}{r|l} & x+2 \\ x^2+1 & x^3+2x^2+3x+2 \\ & x^3 + \quad x \\ \hline & \quad 2x^2+2x+2 \\ & \quad 2x^2 \quad +2 \\ \hline & \quad\quad 2x \end{array}$$

$\therefore\ x^3+2x^2+3x+2 = (x^2+1)(x+2) + 2x$

$\therefore \int \frac{x^3+2x^2+3x+2}{x^2+1}dx = \int \frac{(x^2+1)(x+2)+2x}{(x^2+1)}dx$

$\text{——} = \int x+2+\frac{2x}{x^2+1}\,dx$

$\text{——} = \frac{x^2}{2}+2x+\ln|x^2+1|+C$

$\text{——} = \frac{x^2}{2}+2x+\ln(x^2+1)+C$

$(\text{as } x^2+1>0\ \ \forall x\in\mathbb{R})$

INTEGRATION INVOLVING LOGARITHMIC FUNCTIONS

3 Evaluate:

(a) $\int_0^2 \frac{dx}{x+1}$ (b) $\int_2^4 \frac{3}{4x-2}dx$ (c) $\int_0^2 \frac{2x+1}{x^2+x+1}dx$ (d) $\int_1^2 \frac{2x+1}{2x-1}dx$

a) $\int_0^2 \frac{dx}{x+1} = \left[\ln|x+1|\right]_0^2 = \ln 3 - \ln 1 = \ln 3$

b) $\int_2^4 \frac{3}{4x-2}dx = \frac{3}{4}\int_2^4 \frac{4}{4x-2}dx = \frac{3}{4}\left[\ln|4x-2|\right]_2^4$

$= \frac{3}{4}\left[\ln 14 - \ln 6\right] = \frac{3}{4}\ln\left(\frac{14}{6}\right) = \frac{3}{4}\ln\left(\frac{7}{3}\right)$

c) $\int_0^2 \frac{2x+1}{x^2+x+1}dx = \left[\ln|x^2+x+1|\right]_0^2$

$= \ln(2^2+2+1) - \ln(0^2+0+1)$

$= \ln 7 - \ln 1 = \ln 7$

d) $\int_1^2 \frac{2x+1}{2x-1}dx = \int_1^2 \frac{2x-1+2}{2x-1}dx$

$= \int_1^2 \left[1 + \frac{2}{2x-1}\right]dx$

$= \left[x + \ln|2x-1|\right]_1^2$

$= 2 + \ln|2\times2-1| - (1 + \ln|2\times1-1|)$

$= 2 + \ln 3 - 1 = 1 + \ln 3$

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4 (a) Differentiate $y = \log_e\left(x + \sqrt{x^2 - a^2}\right)$, $x > |a|$ with respect to x.

(b) Hence find $\int \frac{dx}{\sqrt{x^2 - a^2}}$, $x > |a|$.

a) $f(x) = \ln\left(x + \sqrt{x^2 - a^2}\right) \qquad x > |a|$

using Chain rule

$$f'(x) = \frac{1}{x + \sqrt{x^2 - a^2}} \times \left[1 + \frac{1}{2}(x^2 - a^2)^{-1/2} \times 2x\right]$$

$$f'(x) = \frac{1}{x + \sqrt{x^2 - a^2}} \left(1 + (x^2 - a^2)^{-1/2} x\right)$$

$$f'(x) = \left[\frac{1}{x + \sqrt{x^2 - a^2}}\right]\left[1 + \frac{x}{\sqrt{x^2 - a^2}}\right]$$

$$f'(x) = \left[\frac{1}{x + \sqrt{x^2 + a^2}}\right]\left[\frac{\sqrt{x^2 - a^2} + x}{\sqrt{x^2 - a^2}}\right]$$

So $f'(x) = \dfrac{1}{\sqrt{x^2 - a^2}}$

b) $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$$

or $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C$$

with $x > |a|$

5 **(a)** Differentiate $y = \log_e\left(x + \sqrt{x^2 + a^2}\right)$ with respect to x. **(b)** Hence find $\int \frac{dx}{\sqrt{x^2 + a^2}}$.

a) $f(x) = \ln\left(x + \sqrt{x^2 + a^2}\right)$

$f'(x) = \frac{1}{x + \sqrt{x^2 + a^2}} \times \left(1 + \frac{1}{2}(x^2 + a^2)^{-1/2} \times 2x\right)$

$f'(x) = \frac{1}{x + \sqrt{x^2 + a^2}} \times \left(1 + \frac{x}{\sqrt{x^2 + a^2}}\right)$

$f'(x) = \frac{1}{x + \sqrt{a^2 + x^2}} \times \left(\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}\right)$

$\therefore \quad f'(x) = \frac{1}{\sqrt{x^2 + a^2}}$

b) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left|x + \sqrt{x^2 + a^2}\right| + C$

$= \ln\left(x + \sqrt{x^2 + a^2}\right) + C$

as $x + \sqrt{x^2 + a^2} > 0$

as $\sqrt{x^2 + a^2} > x$

6 Use the integrals in questions **4** and **5** to find the following.

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$ (b) $\displaystyle\int \frac{dx}{\sqrt{x^2+1}}$ (c) $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+3}}$

At 4) we established that $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right) + C$ with $x > |a|$

At 5) we established that $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)$ $\forall x \in \mathbb{R}$

a) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \ln\left(x+\sqrt{x^2-1}\right) + C$ with $x > 1$

b) $\displaystyle\int \frac{dx}{\sqrt{x^2+1}} = \ln\left(x+\sqrt{x^2+1}\right) + C$

c) $x^2-4x+3 = (x-2)^2-4+3 = (x-2)^2-1$

$\therefore \displaystyle\int \frac{dx}{\sqrt{x^2-4x+3}} = \int \frac{dx}{\sqrt{(x-2)^2-1^2}}$

We do a change of variable $X = x-2$

$\dfrac{dX}{dx} = 1$ $dX = dx$

$\therefore \displaystyle\int \frac{dx}{\sqrt{x^2-4x+3}} = \int \frac{dX}{\sqrt{X^2-1^2}} = \ln\left(X+\sqrt{X^2-1}\right) + C$ with $X > 1$

$= \ln\left(x-2+\sqrt{x^2-4x+3}\right) + C$

with $x-2 > 1 \iff x > 3$

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6 Use the integrals in questions **4** and **5** to find the following.

(d) $\int \frac{dx}{\sqrt{x^2+6x+13}}$ **(e)** $\int \frac{dx}{\sqrt{x^2-5x+7}}$ **(f)** $\int \frac{dx}{\sqrt{x^2+x+1}}$

d) $x^2+6x+13 = (x+3)^2-9+13 = (x+3)^2+4 = (x+3)^2+2^2$

$\int \frac{dx}{\sqrt{x^2+6x+13}} = \int \frac{dx}{\sqrt{(x+3)^2+2^2}}$ Then we do a change of variable $X = x+3$ $dX = dx$

$\_\_\_ = \int \frac{dX}{\sqrt{X^2+2^2}} = \ln\left(X+\sqrt{X^2+2^2}\right)+C$

$\_\_\_ = \ln\left(x+3+\sqrt{x^2+6x+13}\right)+C$

e) $x^2-5x+7 = \left(x-\frac{5}{2}\right)^2-\frac{25}{4}+7 = \left(x-\frac{5}{2}\right)^2+\frac{3}{4}$

$\int \frac{dx}{\sqrt{x^2-5x+7}} = \int \frac{dx}{\sqrt{\left(x-\frac{5}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$ Then we do a change of variable $X = x-5/2$

$dX = dx$

$\_\_\_ = \int \frac{dX}{\sqrt{X^2+\left(\sqrt{3}/2\right)^2}} = \ln\left(X+\sqrt{X^2+3/4}\right)+C$

$= \ln\left(x-\frac{5}{2}+\sqrt{\left(x-\frac{5}{2}\right)^2+\frac{3}{4}}\right)+C$

f) $x^2+x+1 = \left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

$\int \frac{dx}{\sqrt{x^2+x+1}} = \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}$ Then $X = x+1/2$

$dX = dx$

$\_\_\_ = \int \frac{dX}{\sqrt{X^2+\frac{3}{4}}} = \ln\left(X+\sqrt{X^2+\frac{3}{4}}\right)+C$

$= \ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+C$