INTEGRATION BY PARTS

The product rule for differentiation is associated with the technique known as **integration by parts**, which comes from rearranging the product rule. This rule is useful for solving integrals that cannot be found in easier ways.

If $u(x)$ and $v(x)$ are differentiable functions of x, then the product rule tells you that:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \qquad \text{or} \qquad \frac{d}{dx}(uv) = vu' + uv'$$

Rewriting this: $$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx} \qquad uv' = \frac{d}{dx}(uv) - vu'$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \qquad \int uv'\,dx = uv - \int vu'\,dx$$

$$\int u\,dv = uv - \int v\,du$$

The integrand on the left-hand side is seen to be a product of two expressions involving x: one of these is denoted by u and the other by $\frac{dv}{dx}$ (or just dv). The choice of which expressions to label as u and dv is made so that the integral on the right-hand side can be easily found by normal techniques, such as change of variable (substitution). The arbitrary constant C is inserted into the solution at the appropriate point.

As a general rule, dv should be a function that is easy to integrate and u should be the other function. If they are both easy to integrate, then you should make u the function that will be of a lesser degree (i.e. simpler) after differentiation. This is illustrated in the following examples.

Note that for complex integrals the rule sometimes needs to be applied more than once. You may also need to rearrange terms to solve the desired integral.

Example 23

Find: $\int x\cos x\,dx$

Solution

Let $u = x$, $\frac{dv}{dx} = \cos x$ as this gives $\frac{du}{dx} = 1$, which is easy to work with in the resulting integral.

Thus: $\frac{du}{dx} = 1, \quad v = \sin x$

Integration by parts: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Hence:
$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x + C$$

The constant of integration is added after the last integration is performed.

Alternatively: Another choice of variable could have been $u = \cos x$, $\frac{dv}{dx} = x$.

Thus: $\frac{du}{dx} = -\sin x, \quad v = \frac{x^2}{2}$

Hence: $$\int x\cos x\,dx = \frac{x^2}{2}\cos x - \int \frac{x^2}{2}(-\sin x)\,dx$$

This integral is now more complicated than the original integral.

The method of integration by parts should be used mostly as a last resort, when other known techniques fail. It can be used very effectively to integrate products of different kinds of expressions, for example:

- algebraic and trigonometric functions such as $x\cos x$
- algebraic and logarithmic functions such as $x\log_e x$
- inverse trigonometric functions and logarithmic functions.

Sometimes integration by parts may need to be applied more than once. When there is an integer power of x in the integrand, as in part **(c)** of the example above, then each time it is differentiated the power will be reduced by one until eventually the function becomes a constant. You let u equal this power of x and apply integration by parts successively.

Example 24

Find: $\int \cos^{-1} x\, dx$

Solution

Rewrite as: $\int 1 \times \cos^{-1} x\, dx$

Let $u = \cos^{-1} x$, $\frac{dv}{dx} = 1$. This gives $\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$, $|x| < 1$, and $v = x$.

Integration by parts: $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

Hence: $\int \cos^{-1} x\, dx = x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} dx$

$= x\cos^{-1} x - (1-x^2)^{\frac{1}{2}} + C$ for $|x| < 1$

Example 25

Find **(a)** $\int \log_e x\, dx$ **(b)** $\int xe^x\, dx$ **(c)** $\int x^2 e^x\, dx$

Solution

(a) Let $u = \log_e x$, $\frac{dv}{dx} = 1$. This gives $\frac{du}{dx} = \frac{1}{x}$, $v = x$.

Integration by parts: $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

Hence: $\int \log_e x\, dx = x\log_e x - \int x \times \frac{1}{x} dx$

$= x\log_e x - x + C, \quad x > 0$

$= x\log_e |x| - x + C$

(b) Let $u = x$, $\frac{dv}{dx} = e^x$. This gives $\frac{du}{dx} = 1$, $v = e^x$.

Hence: $\int xe^x\, dx = xe^x - \int e^x\, dx$

$= xe^x - e^x + C$

$= (x-1)e^x + C$

(c) Let $u = x^2$, $\frac{dv}{dx} = e^x$. This gives $\frac{du}{dx} = 2x$, $v = e^x$.

Hence: $\int x^2 e^x\, dx = x^2 e^x - 2\int xe^x\, dx$

You now need to find $\int xe^x\, dx$ by applying integration by parts again, as in part **(b)**:

Let $u = x$, $\frac{dv}{dx} = e^x$. This gives $\frac{du}{dx} = 1$, $v = e^x$.

Hence: $\int xe^x\, dx = xe^x - \int e^x\, dx$

$= xe^x - e^x + C$

$= (x-1)e^x + C$

Thus: $\int x^2 e^x\, dx = x^2 e^x - 2(x-1)e^x + C$

$= (x^2 - 2x + 2)e^x + C$

Example 26

Find: $I = \int e^x \sin x \, dx$

Solution

Let $u = e^x$, $\frac{dv}{dx} = \sin x$. This gives $\frac{du}{dx} = e^x$, $v = -\cos x$.

$$\therefore I = \int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

Now you need to find $\int e^x \cos x \, dx$.

Let $u = e^x$, $\frac{dv}{dx} = \cos x$. This gives $\frac{du}{dx} = e^x$, $v = \sin x$.

$$\therefore \quad \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Thus: $I = -e^x \cos x + e^x \sin x - I$

$$2I = e^x \sin x - e^x \cos x$$

$$\therefore \quad I = \int e^x \sin x \, dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

The arbitrary constant C did not need to be included until the last line.

Example 27

Evaluate: **(a)** $I = \int_0^{\frac{1}{2}} \sin^{-1} x \, dx$ **(b)** $I = \int_0^{\pi} x^2 \sin x \, dx$

Solution

(a) Let $u = \sin^{-1} x$, $\frac{dv}{dx} = 1$. This gives $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$, $v = x$.

Hence:
$$I = \int_0^{\frac{1}{2}} \sin^{-1} x \, dx = \left[x \sin^{-1} x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} \, dx$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - 0 - \left[-(1-x^2)^{\frac{1}{2}}\right]_0^{\frac{1}{2}}$$
$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

(b) Let $u = x^2$, $\frac{dv}{dx} = \sin x$. This gives $\frac{du}{dx} = 2x$, $v = -\cos x$.

Hence:
$$I = \int_0^{\pi} x^2 \sin x \, dx = \left[-x^2 \cos x\right]_0^{\pi} + 2\int_0^{\pi} x \cos x \, dx$$
$$= \pi^2 + 2\int_0^{\pi} x \cos x \, dx$$

Now let $u = x$, $\frac{dv}{dx} = \cos x$. This gives $\frac{du}{dx} = 1$, $v = \sin x$.

Hence:
$$\int_0^{\pi} x \cos x \, dx = [x \sin x]_0^{\pi} - \int_0^{\pi} \sin x \, dx$$
$$= 0 + [\cos x]_0^{\pi}$$
$$= -2$$

$$\therefore \quad I = \int_0^{\pi} x^2 \sin x \, dx = \pi^2 - 4$$

Example 28

Evaluate: **(a)** $I = \int_1^2 x \log_e x \, dx$ **(b)** $I = \int_0^1 \tan^{-1} x \, dx$

Solution

(a) Let $u = \log_e x$, $\frac{dv}{dx} = x$. This gives $\frac{du}{dx} = \frac{1}{x}$, $v = \frac{x^2}{2}$.

$$\therefore I = \int_1^2 x \log_e x \, dx = \left[\frac{x^2}{2}\log_e x\right]_1^2 - \frac{1}{2}\int_1^2 x^2 \times \frac{1}{x} dx$$

$$= 2\log_e 2 - 0 - \frac{1}{2}\int_1^2 x \, dx$$

$$= 2\log_e 2 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2$$

$$= 2\log_e 2 - 1 + \frac{1}{4}$$

$$= 2\log_e 2 - \frac{3}{4}$$

(b) Let $u = \tan^{-1} x$, $\frac{dv}{dx} = 1$. This gives $\frac{du}{dx} = \frac{1}{1+x^2}$, $v = x$.

$$\therefore I = \int_0^1 \tan^{-1} x \, dx = \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$$

$$= \frac{\pi}{4} - 0 - \frac{1}{2}\left[\log_e(1+x^2)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}\log_e 2$$

I. Guidelines for Selecting u and dv:

(There are always exceptions, but these are generally helpful.)

<u>"L-I-A-T-E" Choose 'u' to be the function that comes first in this list:</u>

L: Logrithmic Function

I: Inverse Trig Function

A: Algebraic Function

T: Trig Function

E: Exponential Function

Example A: $\int x^3 \ln x \, dx$

*Since ***lnx*** is a <u>logarithmic function</u> and x^3 is an <u>algebraic function</u>, let:

$u = \ln x$ (L comes before A in LIATE)

$dv = x^3 \, dx$

$du = \frac{1}{x} \, dx$