INTRODUCTION TO DIFFERENTIAL EQUATIONS

1 Verify by differentiation that the function $y = x^n$ is a solution of the differential equation $y' - \frac{n}{x}y = 0$.

$y' = n x^{n-1}$ so $y' - \frac{n}{x} y = n x^{n-1} - \frac{n}{x} \times x^n = n x^{n-1} - n x^{n-1} = 0$

3 Verify the general solution and then specify any parameters in this solution and state the required particular solution of the initial value problem.

(a)

General solution	Differential equation	Initial condition
$y = Ae^{-2x} + 10$	$y' = 2(10 - y)$	$y(0) = 3$

(b)

General solution	Differential equation	Initial condition
$y = Ae^{-x} + 5$	$y' = 5 - y$	$y(0) = 10$

(c)

General solution	Differential equation	Initial condition
$y = \frac{e^x}{A + e^x}$	$y' = y(1 - y)$	$y(0) = 2$

a) $y' = -2Ae^{-2x}$, so $2(10 - y) = 2(10 - Ae^{-2x} - 10)$

$= -2Ae^{-2x} = y'$ indeed

$y(0) = 3$, so $3 = A \times e^0 + 10$ $\boxed{A = -7}$

b) $y' = -Ae^{-x}$ and $5 - y = 5 - (Ae^{-x} + 5) = -Ae^{-x} = y'$ indeed.

$y(0) = A + 5 = 10$ so $\boxed{A = 5}$

c) $y' = \frac{e^x(A + e^x) - e^x(e^x)}{(A + e^x)^2}$ (Quotient rule)

$y' = \frac{Ae^x}{(A + e^x)^2}$

$y(1 - y) = \frac{e^x}{A + e^x}\left(1 - \frac{e^x}{A + e^x}\right) = \frac{e^x}{(A + e^x)}\left(\frac{A + e^x - e^x}{A + e^x}\right) = \frac{Ae^x}{(A + e^x)^2}$

$\therefore y' = y(1 - y)$ $y(0) = \frac{1}{1 + A} = 2$ so $1 = 2 + 2A$

or $2A = -1$

$\boxed{A = -\tfrac{1}{2}}$

4 Verify the general solution and then specify any parameters in this solution to find the required particular solution.

(a)

Trial solution	Differential equation	Initial condition
$y=\frac{-x}{Ax+1}$	$x^2y'=-y^2$	$y(1)=-\frac{1}{3}$

(d)

Trial solution	Differential equation	Initial condition
$y=e^{-x}(ax+b)$	$y''+2y'+y=0$	$y(0)=2, y'(0)=1$

(e)

Trial solution	Differential equation	Initial condition
$y=a\sin 2x+b\cos 2x$	$\frac{d^2y}{dx^2}+4y=0$	$y(0)=1, y'(0)=2$

a) $y'=\frac{(-1)(Ax+1)-Ax(-x)}{(Ax+1)^2}=\frac{-1}{(Ax+1)^2}$

$x^2y'=\frac{-x^2}{(Ax+1)^2}$ whereas $-y^2=\frac{-x^2}{(Ax+1)^2}$ so $x^2y'=-y^2$ indeed.

$y(1)=\frac{-1}{A+1}=-\frac{1}{3}$ so $A+1=3$ $\boxed{A=2}$

d) $y'=-e^{-x}(ax+b)+ae^{-x}=-be^{-x}+ae^{-x}(1-x)$

$y'=e^{-x}(a-ax-b)$

$y''=e^{-x}(-a)-e^{-x}(a-ax-b)=e^{-x}(ax+b-2a)$

$\therefore\ y''+2y'+y=e^{-x}[ax+b-2a+2(a-ax-b)+ax+b]$

$=e^{-x}[x(a-2a+a)+(b-2a+2a-2b+b)]=0$ indeed

$y(0)=b=2$ so $\boxed{b=2}$ $y'(0)=a-b=1$ so $a=1+b=3$ $\boxed{a=3}$

e) $y'=2a\cos 2x-2b\sin 2x$

$y''=-4a\sin 2x-4b\cos 2x$

$y''+4y=-4a\sin 2x-4b\cos 2x+4(a\sin 2x+b\cos 2x)=0$

$y(0)=b=1$ so $\boxed{b=1}$ $y'(0)=2a=2$ so $\boxed{a=1}$

$y=\sin 2x+\cos 2x$

5 Classify each of the following differential equations in terms of its order, degree, dependent variable and independent variable.

(a) $(y')^2 = x^2$ **(b)** $x' = x\sin(t)$ **(c)** $\frac{d^2x}{dt^2} + kx = 0$ **(d)** $\frac{dy}{dx} = y + y^2$

a) 1st order, 2nd degree, dependent variable is y, independent variable is x

b) 1st order, 1st degree, dependent variable is x, independent variable is t

c) 2nd order, 1st degree, dependent variable is x, independent variable is t

d) 1st order, 1st degree, dependent variable is y, independent variable is x

6 Given that $y = e^{kx}$ satisfies the differential equation $\frac{d^2y}{dx^2} = \frac{dy}{dx} + 2y$, find the possible value(s) of k.

$\frac{dy}{dx} = k e^{kx}$ $\quad \frac{d^2y}{dx^2} = k^2 e^{kx}$

$\frac{dy}{dx} + 2y = k e^{kx} + 2 e^{kx} = k^2 e^{kx}$

$\Leftrightarrow k + 2 = k^2 \Leftrightarrow k^2 - k - 2 = 0$

$\Delta = 1 - 4 \times (-2) = 9 = 3^2$

$k = \frac{1-3}{2} = -1$ or $k = \frac{1+3}{2} = 2$

2 possible values of k, -1 and 2

8 Verify by differentiation that the given function is a solution of the corresponding differential equation.

(a) $y=e^{x^n}$ is a solution of $y'-nx^{n-1}y=0$ **(b)** $y=x-x^{-1}$ is a solution of $xy'+y=2x$

(c) $y=\dfrac{x}{1+x}$ is a solution of $y'-(1-y)^2=0$ **(d)** $y=\dfrac{e^{rx}}{1+e^{rx}}$ is a solution of $y'=r(1-y)y$

a) $y' = e^{x^n} \times (n x^{n-1})$

$y' - n x^{n-1} y = n x^{n-1} e^{x^n} - n x^{n-1} \times e^{x^n} = 0$

$\therefore$ $y = e^{x^n}$ is a solution of $y' - n x^{n-1} y = 0$

b) $y = x - x^{-1}$ $\quad y' = 1 - (-1) x^{-2} = 1 + \dfrac{1}{x^2}$

$xy' + y = x\left(1 + \dfrac{1}{x^2}\right) + \left(x - \dfrac{1}{x}\right) = 2x$

$\therefore$ $y = x - x^{-1}$ is a solution of the differential equation $xy' + y = 2x$

c) $y = \dfrac{x}{1+x}$ so $y' = \dfrac{(1+x) - x}{(1+x)^2} = \dfrac{1}{(1+x)^2}$

$y' - (1-y)^2 = \dfrac{1}{(1+x)^2} - \left(1 - \dfrac{x}{(1+x)}\right)^2 = \dfrac{1}{(1+x)^2} - \left(\dfrac{1+x-x}{1+x}\right)^2$

$\qquad = \dfrac{1}{(1+x)^2} - \left(\dfrac{1}{1+x}\right)^2 = 0$

$\therefore$ $y = \dfrac{x}{1+x}$ is a solution to $y' - (1-y)^2 = 0$

d) $y = \dfrac{e^{rx}}{1+e^{rx}}$ $\quad y' = \dfrac{re^{rx}(1+e^{rx}) - re^{rx} \times e^{rx}}{(1+e^{rx})^2} = \dfrac{re^{rx}}{(1+e^{rx})^2}$

$r(1-y)y = r\left(1 - \dfrac{e^{rx}}{1+e^{rx}}\right) \times \dfrac{e^{rx}}{1+e^{rx}} = r\left(\dfrac{1}{1+e^{rx}}\right)\left(\dfrac{e^{rx}}{1+e^{rx}}\right) = \dfrac{re^{rx}}{(1+e^{rx})^2}$

$\therefore$ $y' = r(1-y)y$

and $y = \dfrac{e^{rx}}{1+e^{rx}}$ is a solution to $y' = r(1-y)y$.

9 The amount $m(t)$ of medication remaining in the bloodstream t hours after swallowing a pill can be modelled by the differential equation $\frac{dm}{dt} = -3m + 4e^{-2t}$.

The first term on the RHS represents the rate at which the medication is absorbed from the blood into the body and the second term represents the rate at which the medicine enters the bloodstream. (This is exponential because it is rapid at first, as most of the pill dissolves, then later becomes slower when only a small amount of the pill remains.)

(a) Verify by differentiation that $m(t) = 4(e^{-2t} - e^{-3t})$ is a solution of $\frac{dm}{dt} = -3m + 4e^{-2t}$.

(b) What is the initial amount of medication in the bloodstream?

(c) When is the amount of medication in the bloodstream at it greatest?

(d) What is the long-term amount of medication in the bloodstream?

a) $m(t) = 4(e^{-2t} - e^{-3t})$ $\qquad \frac{dm}{dt} = -8e^{-2t} + 12e^{-3t}$

$-3m + 4e^{-2t} = -3\left[4(e^{-2t} - e^{-3t})\right] + 4e^{-2t}$

$= -12e^{-2t} + 12e^{-3t} + 4e^{-2t} = -8e^{-2t} + 12e^{-3t}$

$\therefore -3m + 4e^{-2t} = \frac{dm}{dt}$ when $m(t) = 4(e^{-2t} - e^{-3t})$

b) At $t = 0$ $\quad m(0) = 4(e^{-2\times 0} - e^{-3\times 0}) = 4(1-1) = 0$

c) The amount of medication in the bloodstream is at its highest when $\frac{dm}{dt} = 0$ or $-8e^{-2t} + 12e^{-3t} = 0$

$\Leftrightarrow 2(e^{-t})^2 = 3(e^{-t})^3 \Leftrightarrow 2 = 3e^{-t}$

$\Leftrightarrow e^{-t} = \frac{2}{3} \Leftrightarrow -t = \ln\frac{2}{3}$ or $t = \ln\left(\frac{3}{2}\right) \sim 0.4055$

~ 24 min

d) $m(t) = 4(e^{-2t} - e^{-3t})$

$\lim_{t \to +\infty} m(t) = \lim_{t \to +\infty} 4(\underbrace{e^{-2t}}_{\to 0} - \underbrace{e^{-3t}}_{\to 0}) = 0$

10 The population of fish in a lake is initially 10 000. The population would increase at a rate of 20% per year, except that there is fishing quota of k fish per year taken from the lake. The population P after t years is modelled by the solution of the differential equation:

$$\frac{dP}{dt} = \frac{1}{5}P - k \text{ with } P = 10\,000 \text{ when } t = 0.$$

Here, k is the constant number of fish removed from the lake each year due to fishing (the fishing quota).

(a) If the fishing quota is set at $k = 1000$ per year, verify by differentiation that the number of fish in the lake t years later will be $P(t) = 5000(1 + e^{\frac{t}{5}})$.

(b) Alternatively, if the fishing quota is set at $k = 3000$ per year, verify by differentiation that the number of fish in the lake t years later will be $P(t) = 5000(3 - e^{\frac{t}{5}})$.

More generally, you can assume an arbitrary but fixed fishing quota of k fish per year.

(c) Verify by differentiation that the number of fish in the lake t years later will be $P(t) = 5(k + (2000 - k)e^{\frac{t}{5}})$.

(d) Hence choose a fishing quota of k fish per year to maintain the fish population at its initial value of 10 000.

a) $P(t) = 5000\left(1 + e^{t/5}\right) = 5000 + 5000\,e^{t/5}$

$\frac{dP}{dt} = 1000\,e^{t/5}$ whereas $\frac{1}{5}P - k = \frac{1}{5} \times 5000\left(1 + e^{t/5}\right) - k$

$= 1000\left(1 + e^{t/5}\right) - k$

$= 1000\,e^{t/5} + 1000 - k.$

So $\frac{dP}{dt} = \frac{1}{5}P - k$ if $k = 1000$. for the 2 quantities to be equal.

b) $P(t) = 5000\left(3 - e^{t/5}\right) = 15{,}000 - 5000\,e^{t/5}$

$\frac{dP}{dt} = -1000\,e^{t/5}$ whereas $\frac{1}{5}P - k = 1000\left(3 - e^{t/5}\right) - k$

$= -1000\,e^{t/5} + 3000 - k$

So for the 2 quantities to be equal, we must have $k = 3000$.

c) if $P(t) = 5\left(k + (2000 - k)\,e^{t/5}\right) = 5k + 10{,}000\,e^{t/5} - 5k\,e^{t/5}$

then $\frac{dP}{dt} = 2000\,e^{t/5} - k\,e^{t/5} = (2000 - k)\,e^{t/5}$

$\frac{1}{5}P - k = k + 2000\,e^{t/5} - k\,e^{t/5} - k = (2000 - k)\,e^{t/5}$

$\therefore \frac{dP}{dt} = \frac{1}{5}P - k$ when $P = 5\left[k + (2000 - k)\,e^{t/5}\right]$

d) For $P = 10{,}000$, we must have $10000 = 5\left[k + (2000 - k)\,e^{t/5}\right]$

$\Leftrightarrow 2000 = k + (2000 - k)\,e^{t/5}$

For this to be true for any k, we must have $k = 2{,}000$.

in that case $\frac{dP}{dt} = (2000 - 2000)\,e^{t/5} = 0$ indeed (no growth)