# FINITE GEOMETRIC SERIES

A **geometric series** is a set of terms in which each term is formed by multiplying the preceding term by a constant number. The series starts with the **first term**, which is usually denoted by $a$. The constant multiplier is called the **common ratio** and is usually denoted by $r$.

The 'sum to $n$ terms' of a geometric series is given by $S_n = a + ar + ar^2 + \dots + ar^{n-1}$.

- The $n$th term of this series is $T_n = ar^{n-1}$
- The common ratio is given by $r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = \dots - \frac{T_n}{T_{n-1}}$
- Also note that $S_n = T_1 + T_2 + T_3 + \dots + T_n$

You call $S_n$ the 'sum to $n$ terms' of the series. This is a finite series.

$S_\infty = T_1 + T_2 + T_3 + \dots + T_n + \dots$ is an infinite geometric series.

Using sigma notation, we have: $S_n = \sum_{k=1}^{n} ar^{k-1}$

Thus $T_1 = a$ and $T_n = ar^{n-1}$.

## Sum of a geometric series

You can obtain a formula for $S_n$ so that you don't always have to add the terms to find the sum.

Write: $S_n = a + ar + ar^2 + \dots + ar^{n-1}$ [1]

Multiply both sides of [1] by $r$: $rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$ [2]

[1] − [2]: $S_n - rS_n = a - ar^n$

$S_n(1 - r) = a(1 - r^n)$

Hence: $S_n = \frac{a(1-r^n)}{1-r}$ for $r < 1$

Alternatively, finding [2] − [1] gives: $S_n = \frac{a(r^n-1)}{r-1}$ for $r > 1$

Note that when $r = 1$, the series becomes $a + a + a + \dots$ to $n$ terms and $S_n = na$.

It is also important to realise that $S_n = S_{n-1} + T_n$, so $T_n = S_n - S_{n-1}$, $n > 1$.

## Summary of formulae

- $a + ar + ar^2 + \dots + ar^{n-1}$ is a geometric series
- $r$ is the common ratio
- $T_1 = a$
- $T_n = ar^{n-1}$
- $S_n = \frac{a(1-r^n)}{1-r}$ for $r < 1$
- $S_n = \frac{a(r^n-1)}{r-1}$ for $r > 1$
- $T_n = S_n - S_{n-1}$, $n > 1$

## Example 22

Find the sum of the first eight terms of the geometric series $3 + 6 + 12 + \ldots$

### Solution

$a = 3, r = 2, n = 8$:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_8 = \frac{3(2^8 - 1)}{2 - 1}$$

$$S_8 = 3(2^8 - 1) = 765$$

## Example 23

For the series $4 + 2 + 1 + \ldots$, find $S_6$ and $S_{10}$.

### Solution

$\frac{T_2}{T_1} = \frac{2}{4} = \frac{1}{2}$ and $\frac{T_3}{T_2} = \frac{1}{2}$, so the series is geometric with $a = 4$ and $r = \frac{1}{2}$.

Now: $S_n = \frac{a(1 - r^n)}{1 - r}$

For $n = 6$:

$$S_6 = \frac{4\left(1 - \frac{1}{2^6}\right)}{1 - \frac{1}{2}}$$

$$S_6 = 8\left(1 - \frac{1}{64}\right) = 8 \times \left(\frac{64 - 1}{64}\right)$$

$$= 8 \times \frac{63}{64} = \frac{63}{8} = 7\frac{7}{8}$$

For $n = 10$:

$$S_{10} = \frac{4\left(1 - \frac{1}{2^{10}}\right)}{1 - \frac{1}{2}}$$

$$S_{10} = 8\left(1 - \frac{1}{1024}\right) = 8 \times \left(\frac{1024 - 1}{1024}\right)$$

$$= 8 \times \frac{1023}{1024} = \frac{1023}{128} = 7\frac{127}{128}$$

These values of $S_n$ for $n = 6$ and $n = 10$ appear to be getting closer to 8.

The graph of $S_n = 8\left(1 - \frac{1}{2^n}\right)$ for $n = 1, 2, 3, \ldots 10$ is shown here.