THE QUADRATIC FORMULA

The equation $ax^2 + bx + c = 0$, $a \neq 0$ is the general quadratic equation.

If you solve this equation by completing the square, you obtain the **quadratic formula**, which is a solution that must be true for all quadratic equations. This formula enables us to solve quadratic equations even when the factors are not obvious.

$$ax^2 + bx + c = 0$$

$$\Leftrightarrow ax^2 + bx = -c$$

$$\Leftrightarrow x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 = -\frac{c}{a}$$

we complete the square for the left-hand side (LHS) of the equation.

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

We use the notation: $\Delta = b^2 - 4ac$ $\quad\Delta$ is called "the discriminant".

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2} \qquad \text{Equation (1)}$$

We note that the LHS of the equation is always positive, as it is a square, therefore there are 3 possibilities, depending of the sign of the discriminant Δ:

1) **if $\Delta < 0$:** there is a positive quantity on the LHS, whereas the quantity of the RHS – which is $\frac{\Delta}{4a^2}$ – is negative. a positive quantity cannot be equal to a negative quantity, therefore there are no x for which $ax^2 + bx + c = 0$; in that case, the equation $ax^2 + bx + c = 0$ has **no solutions.**

2) **if $\Delta = 0$:** in that case, Equation (1) is $\left(x + \frac{b}{2a}\right)^2 = 0$; this is only possible when $x = -\frac{b}{2a}$.

3) **if $\Delta > 0$**: in that case, Equation (1) is $\left(x + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2}$. There are two possible solutions, $x + \frac{b}{2a} = \pm\sqrt{\frac{\Delta}{4a^2}}$ which is equivalent to $x + \frac{b}{2a} = \pm\frac{\sqrt{\Delta}}{2a}$ or $x = -\frac{b}{2a} \pm \frac{\sqrt{\Delta}}{2a}$ or:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} \qquad \text{(with } \Delta = b^2 - 4ac\text{)}$$

This last formula is called "**the quadratic formula**".

THE QUADRATIC FORMULA

Example 16

Use the quadratic formula to solve the following quadratic equations.

(a) $x^2 + 8x + 12 = 0$ **(b)** $x^2 - 3x - 2 = 0$

Solution

(a)

$\Delta = b^2 - 4ac = 8^2 - 4 \times 1 \times 12 = 16$

Δ is positive, therefore the equation has two solutions which are:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-8 \pm \sqrt{16}}{2 \times 1} = \frac{-8 \pm 4}{2}$$

These solutions are:

$$x_1 = \frac{-8+4}{2} = \frac{-4}{2} = -2$$

$$x_2 = \frac{-8-4}{2} = \frac{-12}{2} = -6$$

The quadratic expression can be factorised as:

$$x^2 + 8x + 12 = \big(x - (-2)\big)\big(x - (-6)\big)$$

which simplifies as:

$$x^2 + 8x + 12 = (x+2)(x+6)$$

(b)

$\Delta = b^2 - 4ac = (-3)^2 - 4 \times 1 \times (-2)$
$= 9 + 8 = 17$

Δ is positive, therefore the equation has two solutions which are:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{3 \pm \sqrt{17}}{2 \times 1} = \frac{3 \pm \sqrt{17}}{2}$$

These solutions are:

$$x_1 = \frac{3 + \sqrt{17}}{2}$$

$$x_2 = \frac{3 - \sqrt{17}}{2}$$

The quadratic expression can be factorised as:

$$x^2 - 3x - 2 = \left(x - \frac{3+\sqrt{17}}{2}\right)\left(x - \frac{3-\sqrt{17}}{2}\right)$$

(c) $2x^2 - 4x - 7 = 0$

$\Delta = b^2 - 4ac = 4^2 - 4 \times 2 \times 7 = -40$

Δ is negative, therefore the equation has NO solutions.

(d) $x^2 - 4x + 4 = 0$

$\Delta = b^2 - 4ac = (-4)^2 - 4 \times 1 \times 4 = 16 - 16$
$= 0$

$\Delta = 0$, therefore the equation has one solution which is:

$$x = \frac{-b}{2a} = \frac{4}{2 \times 1} = 2$$

In fact, we say this solution is “double” as in that case, the quadratic expression can be factorised as:

$$x^2 - 4x + 4 = (x-2)(x-2)$$

which simplifies as:

$$x^2 - 4x + 4 = (x-2)^2$$