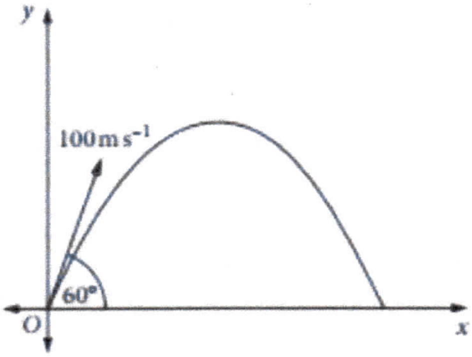
PROJECTILE MOTION

1 A particle is projected with a speed of 100 metres per second from a horizontal plane at an angle of 60°.

Given $\underset{\sim}{v}(t) = 100\cos 60°\underset{\sim}{i} + (100\sin 60° - 10t)\underset{\sim}{j}$, find:

(a) when the particle reaches its greatest height
(b) its position vector $\underset{\sim}{r}(t)$
(c) the greatest height reached
(d) the time of flight
(e) the horizontal range
(f) the equation of the trajectory.



a) The particle reaches its greatest height when $V_y = 0$, i.e when $100\sin 60 - 10t = 0 \iff t = 10\sin 60$ so when $t = 5\sqrt{3}$ s.

b) $\vec{r}(t) = \int (100\cos 60)\vec{i} + (100\sin 60 - 10t)\vec{j}$

$\vec{r}(t) = (50t)\vec{i} + (50\sqrt{3}t - 5t^2)\vec{j} + \vec{C}$. At $t=0$, $\vec{r}(0) = \vec{0}$ so $\vec{C} = \vec{0}$

$\vec{r}(t) = (50t)\vec{i} + (50\sqrt{3}t - 5t^2)\vec{j}$

c) At $t = 5\sqrt{3}$ s (greatest height), $y(5\sqrt{3}) = 50\sqrt{3} \times 5\sqrt{3} - 5(5\sqrt{3})^2$

So $y(5\sqrt{3}) = 750 - 5 \times 75 = 525$ m

d) $y(t) = 0$ when $50\sqrt{3}t - 5t^2 = 0$

$\iff t(10\sqrt{3} - t) = 0$ so at $t = 0$ or $t = 10\sqrt{3}$ s.

which is the time of flight.

e) At $t = 10\sqrt{3}$ s, $x(10\sqrt{3}) = 50 \times 10\sqrt{3} = 500\sqrt{3}$ m

f) $x = 50t$, so $t = x/50$.

$y = 50\sqrt{3}t - 5t^2 = 50\sqrt{3} \times \frac{x}{50} - 5 \times \left(\frac{x}{50}\right)^2 = x\sqrt{3} - \frac{x^2}{500}$

The equation of the trajectory is $y = x\sqrt{3} - \frac{x^2}{500}$

PROJECTILE MOTION

2 A particle is projected with a speed of $80\,\text{m}\,\text{s}^{-1}$ from a horizontal plane at an angle of $30°$.

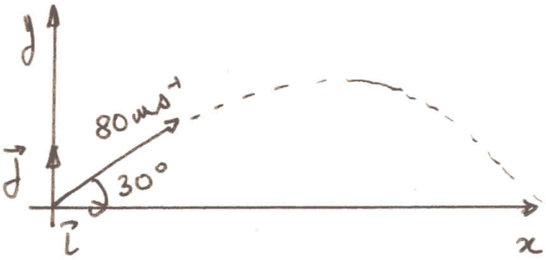
(a) Write the vector for $\underset{\sim}{v}(0)$.

(b) Given that $\underset{\sim}{v}(t) = |\underset{\sim}{v}(0)|\cos\alpha\,\underset{\sim}{i} + (|\underset{\sim}{v}(0)|\sin\alpha - g\,t)\,\underset{\sim}{j}$, find $\underset{\sim}{r}(t)$, using $g = 10\,\text{m}\,\text{s}^{-2}$.

(c) Hence find the time of flight and the range of the projectile.

(d) Find the maximum height reached by the projectile.

(e) Show that the equation of the trajectory is $y = \dfrac{x}{\sqrt{3}} - \dfrac{x^2}{960}$.



a) $\vec{a} = -g\vec{j}$

$\vec{V(0)} = 80\cos 30\,\vec{i} + 80\sin 30\,\vec{j}$

$\vec{V(0)} = \dfrac{80\sqrt{3}}{2}\vec{i} + 80 \times \dfrac{1}{2}\vec{j} = 40\sqrt{3}\,\vec{i} + 40\,\vec{j}$

b) $\vec{V}(t) = |\vec{V(0)}|\cos\alpha\,\vec{i} + \left[|\vec{V(0)}|\sin\alpha - gt\right]\vec{j}$

$\vec{r}(t) = \int \vec{V}(t)\,dt = \left(|\vec{V(0)}|\cos\alpha\, t\right)\vec{i} + \left[|\vec{V_0}|\sin\alpha\, t - \frac{1}{2}gt^2\right]\vec{j} + \vec{C}$

At $t=0$ $\quad \vec{r}(0) = \vec{0}$ $\quad$ so $\quad \vec{C} = \vec{0}$

$\vec{r}(t) = \left[|\vec{V(0)}|\cos\alpha\, t\right]\vec{i} + \left[|\vec{V(0)}|\sin\alpha\, t - 5t^2\right].$

c) $y(t) = 0$ when $t = 0$ or for $t = \dfrac{|\vec{V(0)}|\sin\alpha}{5} = \dfrac{80\sin 30}{5} = 8\,s$

Range is value of $x(8) = |\vec{V(0)}|\cos\alpha \times 8 = 80 \times \cos 30 \times 8 = 320\sqrt{3}\,m$

d) Maximum height when $V_y(t) = 0$, i.e. when $t = \dfrac{|\vec{V(0)}|\sin\alpha}{g}$

i.e. $t = \dfrac{80\sin 30}{10} = 8 \times \dfrac{1}{2} = 4\,s.$

$y(4) = |\vec{V(0)}|\sin\alpha \times 4 - 5 \times 4^2 = 80\sin 30 \times 4 - 80 = 80\,m$

e) $x(t) = |\vec{V(0)}|\cos\alpha\, t = 80\dfrac{\sqrt{3}}{2}t = 40\sqrt{3}\,t.$ $\quad$ so $t = x/40\sqrt{3}$

$y = 80 \times \dfrac{1}{2} \times \left(\dfrac{x}{40\sqrt{3}}\right) - 5\left(\dfrac{x}{40\sqrt{3}}\right)^2$ $\quad$ so $y = \dfrac{x}{\sqrt{3}} - \dfrac{x^2}{960}$

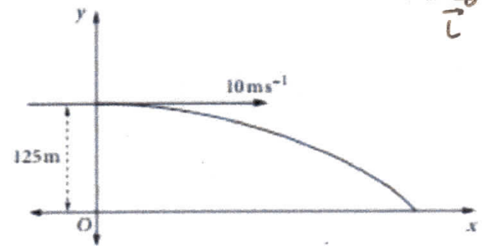
PROJECTILE MOTION

3 An object is projected horizontally from the top of a building 125 m high at a speed of $10\,\text{m s}^{-1}$. Using $g = 10\,\text{m s}^{-2}$, find:

(a) $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$

(b) the time when the object hits the ground and its distance from the base of the building

(c) the maximum height reached by the object.



a) $\vec{a} = -g\vec{j}$

$\vec{v}(t) = (-gt)\vec{j} + \vec{C}$ At $t=0$, $\vec{v}(0) = 10\vec{i}$ so $\vec{C} = 10\vec{i}$

$\vec{v}(t) = 10\vec{i} - gt\,\vec{j} = 10\vec{i} - 10t\,\vec{j}$

$\vec{r}(t) = \int \vec{v}(t)\,dt = (10t)\vec{i} - 5t^2\vec{j} + \vec{K}$.

At $t=0$ $\vec{r}(0) = 125\vec{j}$ so $\vec{K} = 125\vec{j}$

$\vec{r}(t) = [10t]\vec{i} - 5t^2\vec{j} + 125\vec{j} = [10t]\vec{i} + [125 - 5t^2]\vec{j}$

b) $y(t) = 0$ when $125 - 5t^2 = 0$, i.e. $t^2 = \frac{125}{5}$ so $t = 5s$.

when $t = 5$, $x(5) = 10 \times 5 = 50$ m

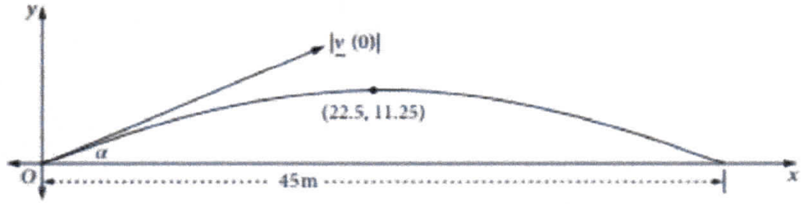
c) it's 125 m, at the point when it starts its journey.

PROJECTILE MOTION

8 A ball is projected so that its horizontal range is 45 metres. It passes through a point 11.25 metres vertically above and 22.5 metres horizontally from the point of projection.

(a) If $|\underset{\sim}{v}(0)|$ is the initial velocity and α the angle of projection, find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(b) Given $g = 10\,\mathrm{m\,s^{-2}}$, find the angle of projection and the speed of projection.



$\vec{j}$ $\vec{i}$

a) $\vec{a} = -g\vec{j}$ so $\vec{V}(t) = \int \vec{a}(t)\,dt = (-gt)\vec{j} + \vec{C}$

At $t=0$ $\vec{V}(0) = |\vec{V}(0)| \cos\alpha\, \vec{i} + |\vec{V}(0)| \sin\alpha\, \vec{j}$.

So $\vec{V}(t) = [|\vec{V}(0)| \cos\alpha]\vec{i} + [-gt + |\vec{V}(0)| \sin\alpha]\vec{j}$

$\vec{r}(t) = \int \vec{V}(t)\,dt = [|\vec{V}(0)| \cos\alpha\, t]\vec{i} + [-\frac{1}{2}gt^2 + |\vec{V}(0)| \sin\alpha\, t]\vec{j} + \vec{K}$.

At $t=0$ $\vec{r}(0) = \vec{0}$, so $\vec{K} = \vec{0}$

$\vec{r}(t) = [|\vec{V}(0)| \cos\alpha t]\vec{i} + [-\frac{1}{2}gt^2 + |\vec{V}(0)| \sin\alpha\, t]\vec{j}$

b) The maximum height is when $t = \frac{|\vec{V}(0)| \sin\alpha}{g} = \frac{|\vec{V}(0)| \sin\alpha}{10}$

At that time, $x = |\vec{V}(0)| \cos\alpha \times \frac{|\vec{V}(0)| \sin\alpha}{10} = \frac{|\vec{V}(0)|^2 \sin 2\alpha}{20} = 22.5$

so $|\vec{V}(0)|^2 \sin 2\alpha = 450 = |\vec{V}(0)|^2 \sin\alpha \cos\alpha = 225$ Equation ①

Further $y_{max} = -5\left(\frac{|\vec{V}(0)| \sin\alpha}{10}\right)^2 + \frac{|\vec{V}(0)|^2 \sin^2\alpha}{10} = \frac{|\vec{V}(0)|^2 \sin^2\alpha}{20} = 11.25$

so $|\vec{V}(0)|^2 \sin^2\alpha = 225$ Equation ②

So, dividing ② by ①, we obtain $\tan\alpha = 1$ so $\alpha = \pi/4$

And therefore $|\vec{V}(0)|^2 = \frac{225}{(\sin \pi/4)^2} = \frac{225}{\frac{2}{4}} = \frac{225}{1/2} = 450$

$\therefore |\vec{V}(0)| = \sqrt{450} = 15\sqrt{2}\ \mathrm{ms^{-1}}$

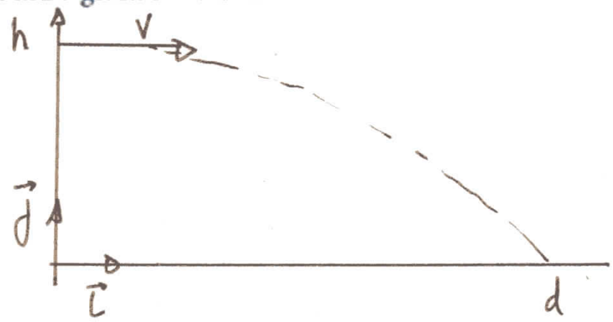
PROJECTILE MOTION

10 A ball is thrown horizontally with speed $v\,\mathrm{m\,s^{-1}}$ from a point h metres above the ground and lands at a horizontal distance d metres from the point of release. Use $g = 10\,\mathrm{m\,s^{-2}}$.

(a) Find expressions for $\underset{\sim}{v}(t)$ and $\underset{\sim}{r}(t)$.

(b) Find v given $d = 3$ and $h = 1.25$.

(c) Find h given $v = 10$ and $d = 20$.



$\vec{a}(t) = -g\vec{j}$

$\vec{v}(t) = \int \vec{a}\,dt = (-gt)\vec{j} + \vec{C}$

At $t=0$ $\vec{v}(0) = v\vec{i}$ so $\vec{C} = v\vec{i}$

$\vec{v}(t) = v\vec{i} + [-gt]\vec{j} = v\vec{i} - 10t\,\vec{j}$

$\vec{r}(t) = \int \vec{v}(t)\,dt = vt\vec{i} - 10\frac{t^2}{2}\vec{j} + \vec{K} = vt\vec{i} - 5t^2\vec{j} + \vec{K}$

At $t=0$ $\vec{r}(0) = h\vec{j}$ so $\vec{K} = h\vec{j}$

$\vec{r}(t) = [vt]\vec{i} + [h - 5t^2]\vec{j}$

b) $y = 0$ when $t^2 = \frac{h}{5} = \frac{1.25}{5} = 0.25$ so $t = 0.5\,s$.

At $t = 0.5\,s$, $x = v \times 0.5 = 3$ so $v = \frac{3}{0.5} = 6\ \mathrm{ms^{-1}}$

c) if $d = 20$, $vt = 20$ or $t = \frac{20}{v} = \frac{20}{10} = 2\,s$.

when $x = 20$, $y = 0$, i.e. $h - 5t^2 = 0$ so $h = 5t^2$

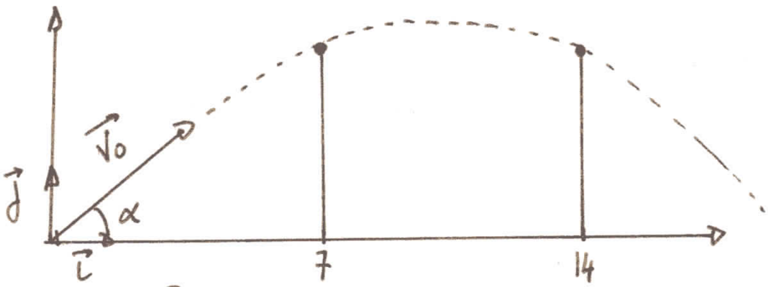
$\therefore h = 5 \times 2^2 = 20\ \mathrm{m}$

PROJECTILE MOTION

16 A particle is projected to just clear two walls that are each 7 m tall, and 7 m and 14 m respectively away from the point of projection. It is given that $\underset{\sim}{r}(t) = Vt\cos\alpha\underset{\sim}{i} + \left(Vt\sin\alpha - 5t^2\right)\underset{\sim}{j}$.

(a) If α is the angle of projection, prove that $\tan\alpha = 1.5$.

(b) Show that if the walls are h metres high and are respectively b metres and c metres distant from the point of projection, then $\tan\alpha = \frac{h(b+c)}{bc}$.



a)

$\vec{r}(t) = Vt\cos\alpha\,\vec{i} + [Vt\sin\alpha - 5t^2]\,\vec{j}$

when $x = 7$, $y = 7$ so $7 = Vt_1\cos\alpha$ ① and $7 = Vt_1\sin\alpha - 5t_1^2$ ②

when $x = 14$, $y = 7$ so $14 = Vt_2\cos\alpha$ ③ and $7 = Vt_2\sin\alpha - 5t_2^2$ ④

$\therefore$ Dividing ② by ①, we obtain $t_2 = 2t_1$

④ becomes: $7 = 2Vt_1\sin\alpha - 20t_1^2$

From ③ $Vt_1\sin\alpha = 7 + 5t_1^2$ so ④ $\Rightarrow 7 = 2[7+5t_1^2] - 20t_1^2$

$\therefore\ 10t_1^2 = 7$ so $t_1 = \sqrt{7/10}$

Hence $\cos\alpha = \dfrac{7}{V\sqrt{7/10}} = \dfrac{\sqrt{70}}{V}$ and $\sin\alpha = \dfrac{7 + 5\times\frac{7}{10}}{V\sqrt{7/10}} = \dfrac{\frac{21}{2}}{V\sqrt{\frac{7}{10}}} = \dfrac{3\sqrt{70}}{2V}$

Hence $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{3\sqrt{70}}{2V} \div \dfrac{\sqrt{70}}{V} = \dfrac{3}{2}$

b) if $b = Vt_1\cos\alpha$ then $t_1 = b/V\cos\alpha$

then $\dfrac{Vb\sin\alpha}{V\cos\alpha} - 5\left(\dfrac{b}{V\cos\alpha}\right)^2 = h$ so $b\tan\alpha - \dfrac{5b^2}{V^2\cos^2\alpha} = h$

Likewise $c\tan\alpha - \dfrac{5c^2}{V^2\cos^2\alpha} = h$

So $bc^2\tan\alpha - b^2c\tan\alpha = c^2h - b^2h$

$\tan\alpha\,[bc(c-b)] = h(c^2-b^2) = h(c-b)(c+b)$

$\therefore\ \tan\alpha = \dfrac{h(b+c)}{bc}$